This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

**Usage guidelines**

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

**About Google Book Search**

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

MATHEMATICAL QUESTIONS

SOLUTIONS

VOL. LXII

Digitized by Google

Math 388.86

SCIENCE CENTER LIBRARY

FROM THE BEQUEST OF

HORACE APPLETON HAVEN,

OF PORTSMOUTH, N. H.

(Class of 1842.)

25 Oct. 1892.

Digitized by Google

Digitized by Google

Digitized by Google

VIII. 339
LIBRARY

# MATHEMATICAL QUESTIONS AND SOLUTIONS,

FROM THE "EDUCATIONAL TIMES,"

WITH MANY

PAPERS AND SOLUTIONS

IN ADDITION TO THOSE

*PUBLISHED IN THE "EDUCATIONAL TIMES,"*

WITH

AN APPENDIX.

EDITED BY

W. J. C. MILLER, B.A.,

REGISTRAR OF THE GENERAL MEDICAL COUNCIL.

VOL. LVII.

LONDON:

FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

1892.

Digitized by Google

~~VIII. 399~~
Math 388.86

Haven Fund.
(57.)

*** Of this series there have now been published fifty-seven Volumes, each of which contains, in addition to the papers and solutions that have appeared in the *Educational Times*, an equal quantity of new articles, and comprises contributions, in all branches of Mathematics, from most of the leading Mathematicians in this and other countries.

New Subscribers may have any of these Volumes at Subscription-prices.

Digitized by Google

# LIST OF CONTRIBUTORS.

AIYAR, Professor RATRAM, M.A.; Trichinopoly.
AIYAR, Professor SWAMINATHA, M.A.; Madras.
ALLEN, Rev. A. J. C., M.A.; Chester College.
ALLMAN, Professor GEORGE J., LL.D.; Galway.
ANDERSON, Prof., M.A.; Queen's Coll., Galway.
ANTHONY, EDWYN, M.A.; The Elms, Hereford.
ARMENANTE, Professor; Pesaro.
ARNOLD, J., M.A.; Institution, Belfast.
BALL, Sir ROBT. STAWELL, LL.D., F.R.S.; Dublin.
BARNIVILLE, JOHN J., M.A.; Belfast.
BARTON, W. J., M.A.; Highgate, London.
BASÚ, Prof. ABINASH CHANDRA, M.A.; Calcutta.
BASÚ, Prof. KALIPADA, M.A.; Dacca College.
BATTAGLINI, Prof. GIUSEPPE; Univ. di Roma.
BAYLISS, GEORGE, M.A.; Kenilworth.
BEEVOR, R. J., B.A.; Regent's Park, London.
BELTRAMI, Professor; University of Pisa.
BERG, Professor F. J. VAN DEN; Delft.
BESANT, W. H., D.Sc., F.R.S.; Cambridge.
BEYENS, Professor IGNACIO, M.A.; Cadiz.
BHATTACHARYA, Prof. MUNINDRANATH, M.A.
BHUT, Professor ATH BIJAH, M.A.; Dacca.
BICKERDIKE, C.; Kippax, Yorkshire.
BIDDLE, D.; Gough Ho., Kingston-on-Thames.
BIRCH, Rev. J. G., M.A.; Bunralty, co. Clare.
BLACKWOOD, ELIZABETH, B.Sc.; Boulogne.
BLYTHE, W. H., B.A.; Egham.
BOLAND, J.; Sand Hill, Gibraltar.
BORDAGE, Prof. EDMOND; Coll. de Nantua.
BOULT, S. H., M.A.; Liverpool.
BOURNE, C. W., M.A.; College, Inverness.
BRIERLEY, MORGAN; Delph, near Oldham.
BRILL, J., B.A.; St. John's Coll., Cambridge.
BROCARD, H., Chef de Bataillon; Valence.
BROOKS, Professor E.; Millersville, Pennsylvania.
BROWN, A. CRUM, D.Sc.; Edinburgh.
BROWN, Prof.; Providence, U.S.A.
BRUNYATE, W. E.; Trinity Coll., Cambridge.
BRYAN, G. H., M.A.; Cambridge.
BRYANT, R., D.Sc., B.A.; London Institute.
BUCK, EDWARD, M.A.; Univ. Coll., Bristol.
BURNSIDE, Professor W. S., M.A.; Univ., Dublin.
BURSTALL, H. J. W.; Sch. of St. John's Coll., Cam.
CAMPBELL, J. E., M.A.; Hertford Coll., Oxford.
CAPEL, H. N., LL.B.; Bedford Square, London.
CARMODY, W. P., B.A.; Clonmel Gram. School.
CARR, G. S., M.A.; 50 Larkhall Rise, Clapham.
CASEY, W. P., M.A.; San Francisco.
CATALAN, Professor; University of Liège.
CAVALLIN, Prof., M.A.; University of Upsala.
CAVE, A. W., B.A.; Magdalen College, Oxford.
CAYLEY, A., F.R.S.; Sadlerian Professor of Mathematics in the University of Cambridge.
CHAKRIVARTI, Prof. BYOM., M.A.; Calcutta.
CHARTRES, R., M.A.; 72, Carter St., Manchester.
CHASE, Prof., LL.D.; Haverford College.
CHRISTIE, R. W. D.; Wavertree Park.
CLARKE, Colonel A. R., C.B., F.R.S.; Redhill.
CLARKE, T. W. K.; Devonshire Buildings, Bath.
COCHEZ, Professor; Paris.
COCKLE, Sir JAMES, M.A., F.R.S.; London.
COHEN, ARTHUR, M.A., Q.C., M.P.; Holland Pk.
COLSON, C. G., M.A.; University of St. Andrews.
CONSTABLE, Rev. W. J., M.A.; Uppingham.
COTTERILL, J. H., M.A.; R. N. Coll., Greenwich.
CRAWFORD, G. E., B.A.; Clifton, Bristol.
CREMONA, Prof. LUIGI; Rome.
CROFTON, Professor M. W., F.R.S.; Dublin.
CROKE, J. O'BYRNE, M.A.; Dublin.
CULLEY, Prof., M.A.; St. David's Coll., Lampeter.
CURJEL, H. W., B.A.; The College, Chester.
CURTIS, Prof., M.A., S.J.; Clongowes Coll., Ireland.
CZUBER, Dr. EMANUEL; Prag, Bohemia.
DANIELS, A. E., B.A.; Cambridge.
DARBOUX, Professor; Paris.
DATA, Prof. PROMATHANATH, M.A.; Calcutta.
DAVIES, D. O. S., M.A.; Univ. Coll., Bangor.
DAVIES, Prof. in Brown Univ., Providence, U.S.A.
DAVIS, R. F., M.A.; West Kensington Park.
DAWSON, H. G., B.A.; Christ Coll., Cambridge.
DAY, Rev. H. G., M.A.; Richmond Terr., Brighton.
DE LONGCHAMPS, Professor; Paris.
DE WACHTER, Prof., M.A.; Schaarbeck.
DEY, Prof. NARENDRA LAL, M.A.; Calcutta.
DICK, G. R., M.A.; Victoria University.
DICKSON, J. D. H., M.A.; Peterhouse, Camb.
DOBBS, W. J., M.A.; Grammar School, Totnes.
DOBSON, T., B.A.; Hexham Grammar School
D'OCAGNE, MAURICE; Pontoise.
DROZ, Prof. ARNOLD, M.A.; Porrentruy, Berne.
DUPAIN, J. C.; Professeur au Lycée d'Angoulême.
EASTON, BELLE, B.Sc.; Lockport, New York.
EDWARD, J., M.A.; Head Mast., Aberdeen Coll.
EDWARDES, DAVID, B.A.; Dublin.
ELLIOTT, E. B., M.A.; Fell. Queen's Coll., Oxon.
ELTON, E. F., M.A.; Wellington College.
EMMERICH, Prof., Ph.D.; Mülheim-am-Ruhr.
EVANS, Professor, M.A.; Lockport, New York.
EVERETT, Prof. J. D., D.C.L.; Qu. Coll., Belfast.
FICKLIN, JOSEPH; Prof. in Univ. of Missouri.
FORTEY, H., M.A.; Clifton, Bristol.
FOSTER, F. W., B.A.; Chelsea.
FOSTER, Prof. G. CAREY, F.R.S.; Univ. Coll., Lond.
FOSTER, W. S., Hoddesdon, Herts.
FRANKLIN, Prof. CHRISTINE LADD, M.A., B.Sc.
GALLATLY, W., M.A.; Highgate, London.
GALLIERS, Rev. T., M.A.; Kirstead Rect., Norwich
GENESE, Prof., M.A.; Univ. Coll., Aberystwith.
GERRANS, H. T., M.A.; Fel. & Tut., Wor. Coll., Oxon
GLAISHER, J. W. L., F.R.S.; Fell. Trin. Coll., Camb.
GOLDENBERG, Professor, M.A.; Moscow.
GORDON, A., B.A.; Barnwood Ho., Gloucester.
GOULD, S. C.; Sec. Press Assoc., Manchester, U.S.A.
GRACE, J. H.; Okell's Farm, Halewood.
GRAHAM, R. A., M.A.; Trinity College, Dublin.
GREENFIELD, Rev. W. J., M.A.; Dulwich College.
GREENHILL, A. H., M.A.; Artillery Coll., W'wich.
GREENSTREET, W. J., B.A.; Marling Sc., Stroud.
GREENWOOD, JAMES M.; Kirksville, Missouri.
GRIFFITHS, D. T., M.A.; Jesus College, Oxford.
GRIFFITHS, G. J., M.A.; Fell. Ch. Coll., Camb.
GRIFFITHS, J., M.A.; Fellow of Jesus Coll., Oxon.
GROSS, W., LL.D.; Bournemouth.
GROVE, W. B., B.A.; Perry Bar, Birmingham.
HADAMARD, Professor, M.A.; Paris.
HAIGH, E., B.A., B.Sc.; King's Sch., Warwick.
HALL, Professor ASAPH, M.A.; Washington.
HAMMOND, J., M.A.; Buckhurst Hill, Essex.
HANUMANTA RAU, Professor, B.A.; Madras.
HARKEMA, C.; University of St. Petersburg.
HARRIS, H. W., B.A.; Trinity College, Dublin.
HART, H.; R. M. Academy, Woolwich.
HAUGHTON, Rev. Dr., F.R.S.; Trin. Coll., Dubl.
HENDERSON, R., B.A.; Queen's Coll., Belfast.
HENDRICKS, Prof., M.A.; Des Moines, Iowa.
HEPPEL, G., M.A.; Grove Park, Chiswick.
HERMAN, R. A., M.A.; Trin. Coll., Cambridge.
HERMITE, CH.; Membre de l'Institut, Paris.
HERVEY, F. R. J., M.A.; Worthing.
HILL, Rev. E., M.A.; St. John's College, Camb.
HIME, Col. H. W. L., R.A.; London.
HINTON, C. H., M.A.; Cheltenham College.
HIRST, Dr. T. A., F.R.S.; London.
HOLT, J. R., M.A.; Trinity College, Dublin.
HOPKINS, Rev. G. H., M.A.; Stratton, Cornwall.
HOPKINSON, J., D.Sc., B.A.; Kensington.
HOROBIN, C. J., B.A.; Homerton College.
HOWSE, G. F.; Balliol College, Oxford.
HUDSON, C. T., LL.D.; Manilla Hall, Clifton.
HUDSON, W. H. H., M.A.; Prof. in King's Coll., Lond.
JACKSON, Miss F. H.; Towson, Baltimore.
JACOBS, H. J.; London Institution, Finsbury.
JENKINS, MORGAN, M.A.; London.
JOHNSON, Prof., M.A.; Annapolis, Maryland.
JOHNSTON, T. P., B.A.; Trin. Coll., Dublin.
JOHNSTON, W. J., M.A.; Univ. Coll., Aberystwith.
JOLLIFFE, A. E., M.A.; Corp. Chr. Coll., Camb.
KAHN, A., B.A.; Mildmay Grove, N.
KATHERN, R. P., M.A.; Fleetwood, Bath.
KENNEDY, D., M.A.; Catholic Univ., Dublin.
KIRKMAN, Rev. T. P., M.A., F.R.S.; Croft Rect.
KITCHIN, Rev. J. L., M.A.; Heavitree, Exeter.
KITTUDGE, LIZZIE A.; Boston, United States
KNISELEY, ALEX.; Columbia City, Indiana.
KNOWLES, R., B.A., L.C.P.; Tottenham.
KOEHLER, J.; Rue St. Jacques, Paris.
KOLBE, Rev. Dr.; St. Mary's Coll., Cape Town.
KRISHNAMACHARRY, Prof., M.A.; Tirupati, India.
LACHLAN, R., M.A.; Vine Cottage, Cambridge.
LAMPE, Prof., Ed. of *Jahrb. der Math.*; Berlin.
LANGLEY, E. M., B.A.; Adelaide Sq., Bedford.
LAVERTY, W. H., M.A.; late Exam. in Univ. Oxford.
LAWRENCE, E. J.; Ex-Fell. Trin. Coll., Camb.
LEGABEKE, Professor; Delft.

LEIDHOLD, R., M.A.; Finsbury Park.
LEMOINE, E.; Rue Littré, Paris.
LEUDESDORF, C., M.A.; Fel. Pembroke Coll., Oxon.
LIAGRE, General; Rue Carotz, Brussels.
LONDON, Rev. H., M.A.; Wimbledon.
LONG, EDWARD, L.C.P.; Brussels.
LONGCHAMPS, Prof. de; Paris.
LOWRY, W. H., M.A.; Blackrock, Dublin.
MCALISTER, DONALD, M.A., D.Sc.; Cambridge.
MCCAY, W. S., M.A.; Fell. Trin. Coll., Dublin.
MCCLELLAND, W. J., B.A.; Prin. of Santry School.
MCCOLL, HUGH, B.A.; Boulogne.
MACDONALD, W. J., M.A.; Edinburgh.
MACFARLANE, Prof. A., D.Sc.; Univ. of Texas.
MCINTOSH, ALEX., B.A.; Bedford Row, London.
MACKENZIE, J. L., B.A.; Free Ch. Coll., Aberdeen.
MCLEOD, J., M.A.; R.M. Academy, Woolwich.
MACMAHON, Capt. P. A.; R. M. Academy.
MCMAHON, Prof., M.A.; Cornell Univ., Ithaca.
MACMURCHY, A., B.A.; Univ. Coll., Toronto.
MADHAVARAO, Prof., M.A.; Vizianagram.
MADISON, ISABEL, B.A.; Cardiff.
MAINPRISE, B. W., R. Naval School, Eltham.
MALET, J. C., M.A., F.R.S.; Cork.
MANNHEIM, M.; Prof. à l'Ecole Polytech., Paris.
MARTIN, ARTEMAS, M.A., Ph.D.; Washington.
MASSEY, W. H.; Twyford, Berks.
MATHEWS, Professor, B.A.; London.
MATZ, Prof., M.A.; King's Mountain, Carolina.
MERRIFIELD, J., LL.D., F.R.A.S.; Plymouth.
MERRIMAN, MANSFIELD, M.A.; Yale College.
MEYER, MARY S.; Girton College, Cambridge.
MILLER, W. J. CLARKE, B.A. (EDITOR); The Paragon, Richmond-on-Thames.
MILNE, Rev. J. J., M.A.; Dulwich.
MINCHIN, G. M., M.A.; Prof. in Cooper's Hill Coll.
MITCHESON, Rev. T., B.A.; Temple Avenue Club.
MITTAG-LEFFLER, Professor; Stockholm.
MONCK, Prof. H. ST., M.A.; Trin. Coll., Dublin.
MOORE, H. K., B.A.; Trin. Coll., Dublin.
MOREL, Professor; Paris.
MORGAN, C., B.A.; Instructor to H.M. Fleet.
MORLEY, Prof., M.A.; Haverford Coll., Pennsyl.
MORRICE, G. G., B.A.; Mecklenburgh Sq., Lond.
MUIR, THOMAS, M.A., F.R.S.E.; Bothwell.
MUKHOPADHYAY, Prof. ASUTOSH, M.A., F.R.S.E
MUKHOPADHYAY, SYAMADAS, M.A.; Chinsurah.
MUNINDRA NATH RAY, Prof. M.A., LL.B.
NASH, Prof.; Registrar of Univ. of Calcutta.
NEUBERG, Professor; University of Liège.
NEWCOMB, Prof. SIMON, M.A.; Washington.
NIXON, R. C. S., M.A.; Royal Acad. Inst., Belfast.
NUTHALL, Col.; Richmond-on-Thames.
O'CONNELL, Major-General P.; Bath.
OPENSHAW, Rev. T. W., M.A.; Clifton, Bristol.
ORCHARD, Prof., M.A., B.Sc.; Hampstead.
OWEN, J. A., B.Sc.; Tennyson St., Liverpool.
PANTON, A. W., M.A.; Fell. of Trin. Coll., Dublin.
PAROLING, Prof.; Cornell Univ., U.S.A.
PEIRCE, Prof. B. O.; Harvard College.
PERRIN, EMILY, B.Sc.; St. John's Wood Park.
PHILLIPS, F. B. W.; Balliol College, Oxford.
PILLAI, Professeur, M.A.; Trichy, Madras.
PLAMENEWSKI, H., M.A.; Dahgestan.
POCKLINGTON, H. C., M.A.; Yorks Coll., Leeds.
POLIGNAC, Prince CAMILLE DE; Berkeley Sq.
POLLEXFEN, H., B.A.; Windermere College.
POOLE, GERTRUDE, B.A.; Cheltenham.
PRESSLAND, A. J., M.A.; Academy, Edinburgh.
PRUDDEN, FRANCES E.; Lockport, New York.
PURSER, FREDK., M.A.; Fell. Trin. Coll., Dublin.
PURSER, Prof. J., M.A.; Queen's College, Belfast.
PUTNAM, K. S., M.A.; Rome, New York.
PYDDOKE, H. W., M.A.; Oxford.
RADHARINSHUAN, Prof.; Maharaj. Coll., Madras.
RAMSEY, A. S., M.A.; Fettes Coll., Edinburgh.
RAWSON, ROBERT; Havant, Hants.
READ, H. J., M.A.; Colonial Office.
REES, E. W., M.A.; Penarth, Cardiff.
REEVES, G. M., M.A.; Lee, Kent.
REYNOLDS, B., M.A.; Notting Hill, London.
RICE, JAMES; Craigie Road, Belfast.
RICHARDS, DAVID, B.A.; Christ's Coll., Brecon.
RICHARDSON, Rev. G., M.A.; Winchester Coll.
ROACH, Rev. T., M.A.; Clifton, Bristol.
ROBERTS, A. D., B.A.; Jesus Coll., Oxford.
ROBERTS, R. A., M.A.; Schol. of Trin. Coll., Dublin
OBERTS, SAMUEL, M.A., F.R.S.; London.
OBERTS, W. R., M.A.; Fell. of Trin. Coll., Dub.
OBSON, H. C., B.A.; Sidney Sussex Coll., Cam.
ROGERS, L. J., M.A.; Yorkshire Coll., Leeds.
ROW, RAMACHANDRA, B.A.; Trichinopoly.
ROY, Prof. KALIPRASANNA, M.A.; Agra.
RUGGERO, SIMONELLI; Università di Roma.
RUSSELL, ALEX., B.A.; Military Coll., Oxford.
RUSSELL, J. W., M.A.; Merton Coll., Oxford.
RUSSELL, R., B.A.; Trinity College, Dublin.
ST. CLAIR, J. C., M.A.; Ryder Street, London.
SALMON, Professor, D.D., F.R.S., Trin. Coll., Dub.
SANDERSON, Rev. T. J., M.A.; Royston, Cambs.
SARADARANJAN RAY, Prof., M.A.; Dacca.
SARKAR, Prof. BENI MADHAV, M.A.; Agra.
SARKAR, Prof. NILKANTHA, M.A.; Calcutta.
SCHEFFER, Professor; Mercersbury Coll., Pa.
SCHOUTE, Prof. P. H.; University, Groningen.
SCOTT, A. W., M.A.; St. David's Coll., Lampeter.
SCOTT, Professor CHARLOTTE A., D.Sc.
SCOTT, R. F., M.A.; Fell. St. John's Coll., Camb.
SEGAR, HUGH W.; Trinity Coll., Cambridge.
SEN, Prof. RAJ MOHAN; Rajshabye Coll., Bengal.
SEWELL, Rev. H., B.A.; Wrexham.
SEYMOUR, W. R., M.A.; Tunbridge.
SHARPE, J. W., M.A.; The Charterhouse.
SHARPE, Rev. H. T., M.A.; Cherry Marham.
SHEPHERD, Rev. A. J. P., B.A.; Fell, Q. Coll., Oxf
SHIELDS, Prof., M.A.; Coopwood, Missouri.
SIMMONS, Rev. T. C., M.A.; Grimsby.
SIRCOM, Prof., M.A.; Stonyhurst College.
SITARAMAIYAR, Prof., Hindu Coll., Tinnevelly.
SIVERLY, WALTER; Oil City, Pennsylvania.
SKRIMSHIRE, Rev. E., M.A.; Llandaff.
SMITH, C., M.A.; Sidney Sussex Coll., Camb.
SOPER, H. E., B.A.; Highgate.
STEGGALL, Prof. J. E. A., M.A.; Dundee.
STEEDE, B. H., B.A.; Trin. Coll., Dublin.
STEPHEN, ST. JOHN, B.A.; Caius Coll., Cambridge.
STEWART, H., M.A.; Framlingham, Suffolk.
STORR, G. G., B.A.; Clerk of the Medical Council.
STOTT, WALTER, B.A.; Royal Ins. Co., Liverpl.
SYLVESTER, J. J., D.C.L., F.R.S.; Professor of Mathematics in the University of Oxford.
SYMONS, E. W., M.A.; Fell. St. John's Coll., Oxon.
TAIT, Prof. P. G., M.A.; Univ., Edinburgh.
TANNER, Prof. H. W. L., M.A.; S. Wales Univ. Coll.
TARLETON, F. A., M.A.; Fell. Trin. Coll., Dub.
TATE, JAMES, M.A.; Kedleston Road, Derby.
TAYLOR, Rev. CHARLES, D.D.; Master of St. John's College, Cambridge.
TAYLOR, H. M., M.A.; Fell. Trin. Coll., Camb.
TAYLOR, W. W., M.A.; Ripon Grammar School.
TEBAY, SEPTIMUS, B.A.; Farnworth, Bolton.
TERRY, Rev. T. R., M.A., Ilsley Rectory.
THOMAS, A. E., M.A., Merton College, Oxford.
THOMAS, Rev. D., M.A.; Garsington Rect., Oxford.
THOMPSON, C. H.; St. David's Coll., Lampeter.
THOMSON, Rev. F. D., M.A.; Ex-Fel. St. J. Coll., Cam.
TIRELLI, Dr. FRANCESCO; Univ. di Roma.
TRAILL, ANTHONY, M.A., M.D.; Fell. T.C.D.
TUCKER, R., M.A.; University Coll., London.
VIGARIÉ, EMILE; Laissac, Aveyron.
VINCENZO, JACOBINI; Università di Roma.
VOSE, Professor G. B.; Washington.
WALENN, W. H.; Mem. Phys. Society, London.
WALKER, J. J., M.A., F.R.S.; Hampstead.
WALMSLEY, J., B.A.; Eccles, Manchester.
WARBURTON-WHITE, R., B.A., Salisbury.
WARREN, A. T., M.A.; Usk Terrace, Brecon.
WARREN, R., M.A.; Trinity College, Dublin.
WATHERSTON, Rev. A. L., M.A.; Bowdon.
WATSON, D., M.A.; Folkestone.
WATSON, Rev. H. W.; Ex-Fell. Trin. Coll., Camb.
WELLACOT, Rev. W. T., M.A.; Newton Abbott.
WERTSCH, FRANZ; Weimar.
WHALLEY, L. J., B.Sc.; Leytonstone.
WHITE, J. R., B.A.; Worcester Coll., Oxford.
WHITE, Rev. J., M.A.; R. Naval School, Eltham.
WHITESIDE, G., M.A.; Eccleston, Lancashire.
WHITWORTH, Rev. W. A., M.A.; London.
WILLIAMS, C. E., M.A.; Wellington College.
WILLIAMS, D. J., M.A.; Worcester Coll., Oxon.
WILLIAMSON, B., M.A.; F. & T. Trin. Coll., Dub.
WILLMOT, Rev. W. T., M.A.; Newton Abbott.
WILSON, Ven. Archdeacon J. M., M.A., F.G.S.
WILSON, Rev. J., M.A.; Rect. Bannockburn Acad.
WILSON, Rev. J. R., M.A.; Royston, Cambs.
WOODALL, H. J.; Stockport.
WOODCOCK, T., B.A.; Twickenham.
WOOLHOUSE, W. S. B., F.R.A.S., &c.; London.
YOUNG, JOHN, B.A.; Portadown, Ireland.
ZEEB, Professor, M.A.; Staunton, Virginia.

# CONTENTS.

## Solved Questions.

Page

1069. (The late Professor Wolstenholme, D.Sc.)—Prove that (1) the mean value of $(xyz)^{\frac{1}{2}}$, where $x$, $y$, $z$ are areal coordinates of a point within the triangle of reference, is $\frac{4\pi}{150}$; (2) the mean value of $(wxyz)^{\frac{1}{2}}$, where $w$, $x$, $y$, $z$ are tetrahedral coordinates of a point within the tetrahedron of reference is $\frac{\pi^2}{320}$; and (3) the mean value of $(wxyz)^{n-1}$ is $6\{\Gamma(n)\}^4 \div \Gamma(4n)$. ..... 41

1850. (Professor Sylvester.)—In a parabolic segment cut off by a line perpendicular to the axis two points are taken; find the mean value of such of the chords drawn through them as cut the segment in two points..... ..... 64

2039. (The late Prof. Wolstenholme.) — A triangle is inscribed in a given circle whose radius is $r$: prove that, if all positions of the angular points be equally probable, the mean value of the perimeter is $120/\pi$, and that the mean value of the radius of the inscribed circle is $r(12/\pi^2 - 1)$. ..... 123

2041. (The late Professor Wolstenholme, D.Sc.)—If the whole perimeter ($2a$) and one side ($c$) of a triangle are given, prove that (1) the mean value of its area is $\frac{1}{8}\pi c[a(a-c)]^{\frac{1}{2}}$; and (2) the mean value of this mean value, $c$ being equally likely to have any value from 0 to $a$, is $\frac{\pi}{30}a^2$. 63

2442. (The late Professor Townsend, F.R.S.)—Through a given point in a plane draw a line the sum of the squares of whose distances from any number of given points in the plane shall be a maximum, a minimum, or given. ..... ..... 92

2885. (Artemas Martin, LL.D.)—Three horses are tethered in an equilateral triangular lot, each being fastened to a corner of the lot by a rope $r$ feet long, and there are, in the centre of the lot, $a$ square feet common to all three of the horses. Required the dimensions of the lot. ..... 65

3964. (The late M. Collins, M.A.)—Show that the Chaldean Saros, or period of 223 lunations (= 6585·3212 days), cannot be safely employed oftener than 49 or 50 times successively in predicting lunar eclipses... 92



Digitized by Google

4015. (Dr. C. Taylor.) — Prove that any chord of a rectangular hyperbola subtends equal or supplementary angles at the ends of a perpendicular chord ..... 70

4609. (Professor Christine Ladd, B.Sc.) — A circle of radius R is completely surrounded by $n$ circles. These are in turn surrounded by successive rows of circles, each circle touching the two adjacent circles of its own row, and also two adjacent circles of the next inner row. Required the radius of the circles of the $m^{th}$ row. ..... 71

4964. (Rev. W. A. Whitworth, M.A.)—A regular "screw-polygon" is formed of equal straight lines. Any two adjacent sides contain an angle $\pi - \theta$, and the planes containing any side and either of the adjacent sides contain an angle $\psi$. Show that, if there be $n$ sides in a convolution,

$$\cos \pi/n = \cos \tfrac{1}{2}\theta \cos \tfrac{1}{2}\psi.$$ ..... 71

4969. (R. A. Roberts, M.A.)—If, from any point of a Cassinian oval, tangents be drawn to the curve, prove that their six points of contact lie on a conic. ..... 88

4979. (Professor Hudson, M.A.)—If a conical cup be filled with liquid, prove that the mean pressure at a point in the volume of the liquid is to the mean pressure at a point in the surface of the cup as 4 : 3.... 85

5132. (Professor Crofton, F.R.S.) — From any point P within a triangle lines are drawn parallel to the two sides, meeting the base on D, E. Show (1) (as may be done without integration) that the mean value of the triangle PDE is one-sixth of the former triangle; and find (2) the mean value of the segment DE. ..... 96

5136. (Professor Hendricks, M.A.)—If four bricks are placed on each other at random, with their longest axes horizontal and in the same vertical plane, show that the probability that the pile will stand is $\frac{200}{2304}$. ..... 65

5258. (The late T. Cotterill, M.A.) — If a variable conic through four given points $a$, $b$, $c$, $d$, cut two given lines through $d$ in $p$ and $q$, prove that the six centres of perspective of the pairs of points $p$, $q$, and the three pairs of points taken from the points, $a$ $b$, $c$, describe straight lines. ..... 73

5321. (Professor Hudson, M.A.)—If the path of a ray cut at a constant angle $\alpha$ the surfaces of equal density in a variable medium, prove that $\mu = \mu_0 \epsilon^{\phi \tan \alpha}$, where $\phi$ is the inclination of the path to a fixed line. ..... 33

5322. (J. L. McKenzie, B.A.)—A triangle ABC is inscribed in a conic; through C a line is drawn parallel to the tangent at A, and meeting the conic in D; through A a line is drawn parallel to BC, and meeting the conic in E; prove that DE is parallel to AB..... 52

5345. (Dr. Hart.)—Find a triangle whose sides, perpendicular, line bisecting the vertical angle, and area, shall be integers. ..... 32

5368. (H. W. Harris, B.A.)—Find the locus of the middle point of a chord of an ellipse, making any given angle with the tangent at its extremity; and point out some singularities of this curve. ..... 69

Digitized by Google

6376. (Prof. Minchin, M.A.)—A lake, the area of whose surface is $a$ square miles, and whose temperature is 32° F., is frozen to a depth of $n$ inches in $d$ days. Calculate, in horse-power, the mean rate at which the work of freezing is done, using the well-known numerical values of the other quantities required. ..... 110

7560. (Prof. Asutosh Mukhopadhyay, M.A.)—Two points are taken at random on the surface of a sphere, of radius $\rho$, on the opposite sides of a fixed great circle; prove that the average distance between them is $\rho\left(\pi - \frac{4}{\pi}\right)$. ..... 122

7826. (Prof. Orchard, B.Sc., M.A.)—P, Q, R are three points in a plane; PQ = QR, and R is fixed. P, Q rotate uniformly round Q, R respectively, and PQ, QR separate from a fixed common chord at equal rates and in opposite directions. Show that the motion of P relative to R is oscillatory..... 125

8013. (Rev. T. P. Kirkman, M.A., F.R.S.)—

"Full oft ye have had your fiddlers' fling,
For your own fun over the wine;
And now," quoth Cole, the merry old king,
"Ye shall have it again for mine.
My realm prepares for a week of joy,
At the coming of age of a princely boy.
Of the grand six days' procession in square,
In all your splendour dressed,
Filling the city with music rare
From fiddles five abreast,

"Ye know; but more—swear all amain
In all your fives as ye love to do,
'Ye four sots, never I fiddle again,
This week abreast with any of you.'

"Swear it, or not; that law, of my will,
Must rule the array for every day;
And now ye have three months' warning.
The thirty fives, your squares to fill,
Be ready that Monday morning.

"Fail ye—by all that overawes
The soul of fiddling knave,
My barber from your guzzling jaws
Shall ears full fifty shave."

No little sport
Had city and court
With fiddlers in their fives,
And tales in the air
Of their beaten despair
To the last in fight; for it never came right
With the closing Saturday square.

But, when the big moment arrives,
Forth march those twenty-five fiddlers gay,
While betting hundreds book the array,
Some with hopes and some with fears,
To see that gory waste of ears.

More merry and more they parade and play
From morn to eve, till Saturday
Proclaims the monarch's riddle guessed;
For never a pair had twice abreast
Composed the grand display.

And how did they manage to save their ears?
By paying down every rascal a year's
Full wage to the Royal Astrologer—
Solemn old fox, as learned as old:
None in the land so famed for his knowledge, or
Half so greedy of gold.

He did it but once: win nobler fame
By doing it Q times, counting his way,
Thus—keeping the Monday fives ever the same,
Q times you add five squares, one for each day,
Q times keeping whole the rule of King Cole.

Pray let me ask you for this maximum Q,
And for 5Q and one squares, such that twain
Shall be none, that a common five contain;
For surely you, that went to college, are
More than a match for the old Astrologer.

Do you lack a key? I beg you to see
Cam. and Dub. Math. J., N.S., Vol. V.,
Page 256 of a note by me.
..... 30

8145. (Rev. T. R. Terry, M.A.)—If $x^2+y = 18$, and $x+y^2 = 8$, show that all the values of $x$ are real; find roughly their position; and calculate the greatest value of $x$ to six places of decimals. ..... 93

Digitized by Google

8225. (D. Biddle.)—A sphere 12 inches in diameter rolls down a roof which is inclined at 30° to the horizon, the eaves being 20 feet from the ground; 10 feet in advance of the vertical line thus indicated, and 3 feet from the ground, is the upper and further edge of a horizontal bar of square section. Disregarding friction and the resistance of the air, find how far the sphere must have rolled on the roof, from a position of rest, in order to clear the bar..... 119

8274. (Professor Haughton, F.R.S.) — Owen Harte has performed the astonishing feat (1885) of throwing the 16 lb. hammer the distance of 117 ft. 3½ in. Calculate in foot-pounds the work done by his muscles in this effort. ..... 85

8403. (Prof. Mukhopadhyay, M.A., F.R.A.S.) — "Observations on the motions of the sun-spots have also established the fact that the sun is not strictly a fixed body around which the earth revolves, but that it has a motion of its own through space." (HUXLEY'S *Physiography*, p. 365.) Explain how the above fact is determined by observations of sun-spots. ..... 46

8509. (Professor Sylvester.) — Show that the discriminant of the form $ax^5 + b\lambda\, x^4y + c\lambda^2x^3y^2 + c\mu^2x^2y^3 + b\mu xy^4 + ay^5$ will be a rational integral function of the quantities $a$, $b$, $c$, $\lambda\mu$, $\lambda^5 + \mu^5$, and of the second degree only in respect of the last of them. ..... 54

8515. (By Professor Hudson, M.A.)—Find the area included between an epicycloid and a hypocycloid with the same number $n$ of coincident cusps. ..... 110

8553. (Professor Mahendra Nath Ray, M.A., LL.B.) — If $x$, $y$, $z$ denote the distances of the orthocentre from the angular points of the triangle ABC, prove that

$$xyz\,(ax + by + cz)^3 + abc\,(x^2 + y^2 + z^2)(ax + by + cz)^2 - 4a^3b^3c^3 = 0 \ldots. \quad 48$$

8675. (By Professor Haughton, F.R.S.) — The daily energy of the average diet of all the armies of Europe is estimated at 3694 foot-tons; the daily work done for this expenditure of food (including the work done in moving the man's own body) is estimated at 430·6 foot-tons. If man be regarded as a perfect heat-engine, whose upper temperature is 100° F. (blood-heat), calculate from the foregoing data the lower temperature at which the engine is worked. ..... 83

8842. (D. Edwardes.)—Prove that

$$\int_0^1 \int_0^{\frac{1}{2}\pi} \frac{\log(1 - k\sin^2\phi)}{(1 - k\sin^2\phi)^{\frac{1}{2}}}\, dk\, d\phi = 4\log 2 - 4. \ldots\ldots 77$$

8919. (W. J. Greenstreet, M.A.)—In the ambiguous case, if A be given angle, $a$, $c$ given sides ($c > a$), $C_1$, $C_2$ the two positions of the angular point opposite $c$, show that the radius of nine-point circle of $\Delta BC_1C_2$ is half that of nine-point circle of $\Delta ABC_2$ if $c = 2a$, and is equal to radius of inscribed circle of $BC_1C_2$ if $\sin A = \frac{1}{4}\sqrt{3}$. ..... 62

9115. (Professor Ignacio Beyens.)—Si D est le diamètre de la sphère circonscrite à un tétraèdre equifacial, $\alpha$, $\beta$, $\gamma$ les angles sous lesquels on voit du centre de la sphère des arêtes du tétraèdre, et V le volume du tétraèdre; on aura $V = \frac{1}{3}D^3 \cos\frac{1}{2}\alpha \cos\frac{1}{2}\beta \cos\frac{1}{2}\gamma$. ..... 47

Digitized by Google

9203 (Professor Nash, M.A.)—Prove that the two circles

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = (a\alpha + b\beta + c\gamma)(l\alpha + m\beta + n\gamma),$$
$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = (a\alpha + b\beta + c\gamma)(l'\alpha + m'\beta + n'\gamma)$$

will be orthogonal if

$$1 - \Sigma l \cos A - \Sigma l' \cos A + \Sigma ll' - \Sigma\,(mn' + m'n)\cos A = 0,$$

and therefore the condition that (1) may be orthogonal to the circumcircle is $\Sigma l \cos A = 1$..... 124

9285. (W. J. Greenstreet, B.A.)—Let P be an angular point of a cube, and let the three faces meeting in P touch the in-sphere in A, B, C; find $r$ of small circle through A, B, C, and the sides and angles of the spherical triangle ABC. ..... 80

9299. (A. Gordon.)—If a polygon of $n$ sides be inscribed in a circle, with vertices $A_1A_2A_3 \ldots A_n$ (taken in order), prove that the following conditions must be satisfied:—

$$(A_rA_s)(A_tA_1) + (A_1A_r)(A_sA_t) = (A_rA_t)(A_sA_1) \qquad \text{(Euc., Bk. vi.)}$$
$$(A_rA_s)(A_1A_t)(A_1A_u) + (A_sA_t)(A_1A_u)(A_1A_r) + (A_tA_u)(A_1A_r)(A_1A_s) = (A_rA_u)(A_1A_s)(A_1A_t),$$
$$A_rA_s(A_1A_t)(A_1A_u)(A_1A_v) + (A_sA_t)(A_1A_u)(A_1A_s)(A_1A_r) + (A_tA_u)(A_1A_r)(A_1A_s)(A_1A_v) + (A_uA_v)(A_1A_r)(A_1A_s)(A_1A_t) = (A_rA_v)(A_1A_s)(A_1A\ )(A_1A_u),$$
$$\text{\&c.} \qquad \text{\&c.} \qquad \ldots\ldots 61$$

9755. (Rev. T. C. Simmons, M.A.)—If the base and vertical angle A of a triangle are given, show that the equation to the locus of the positive Brocard-point is $r^4 - 4Rr^3 \cos A \sin\theta + 4r^2R^2(3\sin^2\theta - \sin^2 A) - 16rR^3 \cos A \sin^3\theta + 16R^4 \sin^4\theta = 0$, where B is taken for origin, BC for initial line, and R for the given circum-radius. ..... 70

9838. (Professor Hudson.)—A given segment of a parabola, whose base is a chord perpendicular to its axis, has another such segment inscribed in it, their bases being parallel; find when the inscribed segment is maximum. If, in this last, another maximum segment be inscribed, and so on, show that, when these maximum segments become indefinitely small, they will lie at the centre of gravity of the original segment.... 44

9977. (Professor Lampe, LL.D.)—Investigate formulæ for the sums of the powers of the rational numbers, and, with the notation

$$S^k(x) = 1 + 2^k + \ldots + x^k,$$

prove, from general expressions, that

$$16S_1^5 = S_5 + 10S_7 + 5S_9, \quad 12S_2^3 = S_4 + 7S_6 + 4S_8, \quad 12S_1S_5 = -S_3 + 5S_5 + 8S_7,$$
$$30S_2S_4 = -S_3 + 15S_5 + 16S_7, \quad 72S_1S_2S_3 = 3S_4 + 42S_6 + 27S_8. \ldots 58$$

10064. (Professor Bordage.) — Examine the surface that has for equation $e^{-z} = \cos x \cos y$, and find its lines of curvature..... 89

Digitized by Google

10231. (Professor De Wachter.)—If $a$, $b$, $c$ be the radii of gyration of a given plane area about three axes perpendicular to its plane in A, B, C, show that, if circles be drawn with centres A, B, C, and radii $a$, $b$, $c$, respectively, their radical centre will be the area's centroid..... 91

10253. (H. J. Woodall. Suggested by Quest. 10179.) — Solve (1) the cubic $x^3 + ax = b$, (2) the quintic $x^5 + ax^2 = b$, by means of a continued fraction of the form $x = K/(L + \sqrt[2]{}/\&c.)$; and find (3) whether the form is suitable for calculation. ..... 46

10258. (Professor Neuberg.)—On considère toutes les coniques ayant même axe focal $2a$, un foyer fixe F, et le second foyer F′ en un point quelconque d'une droite donnée. Trouver (1) le lieu des sommets de l'axe focal; (2) l'enveloppe des directrices; (3) l'enveloppe de ces coniques. ..... 74

10267. (Professor Hermite.) — On sait que les coefficients de la puissance $m$ du binôme sont divisibles par $m$, à l'exception du premier et du dernier, lorsque $m$ est un nombre premier. (1) On propose d'étendre cette proposition en montrant que l'expression (A)

$$\{m(m-1)\dots(m-n+1)\}/n!$$

est divisible par $m/\delta$, $\delta$ désignant le plus grand commun diviseur de $m$ et $n$. (2) Soit $\epsilon$ le plus grand commun diviseur des nombres $m+1$ et $n$; on demande d'établir que le nombre entier (A) est divisible par $(m-n+1)/\epsilon$. ..... 91

10334. (F. R. J. Hervey.)—A pack of $N = mn$ cards is dealt into $n$ packs, which are collected in the order of dealing, placing the 1st on the 2nd, these on the 3rd, and so on; and the process repeated with the same or different values of $n$. Show that (1) deals with different values of $n$ are convertible in order; and hence (2) the cycles in which the cards are interchanged by any deal are independent of preceding deals. [For instance, 36 cards, in 4 similarly numbered suits, may be so arranged that, being dealt sometimes into 4, sometimes into 6, the first never disarranges the order of suits, nor the second the order of pips.] (3) Give the conditions which admit of any card being brought to a given place; and (4), this being possible, show, in the most general case, how to assign numbers $p$, $q$ ... whose product = N, so that it may not be necessary to use more than $p-1$ deals of one sort, $q-1$ of another, and so on. ... 55

10503. (Alpha.) — Find the equation of an ellipse passing through the five points $(x_1y_1)$, $(x_2y_2)$, $(x_3y_3)$, $(x_4y_4)$, $(x_5y_5)$. ..... 46

10766. (Professor Morley, M.A.) — In an infinite mass of perfect fluid are three straight parallel vortex-filaments, of unequal strengths. Show that, if the cross section is once an equilateral triangle, it is always an equal equilateral triangle. ..... 56

10794. (Professor Morley, M.A.)—In an infinite mass of perfect fluid are three straight parallel vortex-filaments of unequal strengths. Show that the circular cylinder which passes through the filaments is orthogonal to a fixed circular cylinder. ..... 56

Digitized by Google

10806. (The late Professor Wolstenholme, Sc.D.) — Prove that, if $x, y, z$ be three finite quantities whose sum is zero, and if

$(ax+by+cz)^2 = 4\,(bc+ca+ab)(yz+zx+xy)$ and $ax^2+by^2+cz^2 = 0$,

then will $(a+b+c)(bc+ca+ab) = 9abc$. ..... 38

10859. (W. J. Greenstreet, M.A.)—$\phi\,(x)$ is a positive function of $x$, which does not increase between the values $a$ and $\infty$ of the variable; given $\int_a^\infty f(x)\,dx$ is finite, show that $\int_a^\infty f(x)\,\phi\,(x)\,dx$ is finite. ..... 60

10863. (Rev. T. Roach, M.A.)—Parabolas are drawn, each touching two sides of an equilateral triangle, where they meet the third side; prove that (1) each pair meet on the axis of the third parabola; and (2) the square of the common chord is to the square of the common tangent in the ratio of 16 : 27. ..... 40

10879. (Professor Genese, M.A.)—With any point O in the plane of an ellipse as centre, prove that (1) two real circles can be drawn, in either of which triangles may be inscribed whose sides touch the ellipse; and (2) the radius R of either is given by

$(R^2-OS^2)(R^2-OH^2) = 4b^2R^2$. ..... 37

10894. (Prof. Bourrienne.) — Soient XOY un angle fixe et A un point fixe pris sur OX. On trace un cercle quelconque C tangent à OX et en un point D à OY; puis de A on mène à ce cercle une seconde tangente qui le touche en E. Démontrer que la droite DE passe par un point fixe. ..... 60

10930. (Professor Curtis, S.J., M.A.)—Prove that six conics may be drawn osculating each other at a given point, and each osculating a given conic. ..... 35

10955. (Professor Zerr.) — Find the centroid of the area between the cardioid $\gamma = \frac{1}{2}a\,(1+\cos\theta)$ and its first pedal, when the density at a point varies as the $n^{\text{th}}$ power of the distance of the point from the origin. ..... 43

11002. (Professor de Longchamps.) — On donne deux points fixes O, A (OA = R). Soit $\Delta$ une droite mobile; on abaisse, des points O et A, sur $\Delta$, des perpendiculaires OO′, AA′. Ayant posé $OO' = u$, $AA' = v$, on suppose que la relation $(v-u)^2 = Ru$ est vérifiée. (1) Trouver l'enveloppe de $\Delta$. Cette enveloppe est une courbe Γ, unicursale du sixième ordre et de la quatrième classe, possédant quatre points de rebroussement. En désignant pas $\phi$ l'angle $O'Ox$, on démontrera que les coordonnées d'un point de cette courbe correspondent aux formules

$$x = R\cos\phi\,(1+\sin^2\phi), \quad y = -R\sin\phi\cos^2\phi.$$

(2) Déduire de ces formules les propriétés essentielles de cette courbe. ..... 48

11009. (The late Professor Clifford, F.R.S.)—A bubble consists of $v$ cubic centimetres of water, and requires an amount of work $w$ to increase its radius from 10 to 20 centimetres; if a cubic centimetre of water

Digitized by Google

requires the work $h$ to convert it into steam at the same temperature, find the thickness of the film when the liquid may be regarded as just passing into the gaseous form, the bubble being blown out into a state of extreme tenuity. ..... 42

11018. (D. Biddle.)—In reference to reciprocals, prove that

$$\Sigma_0^{\infty} (1-a/A)^n = A/a, \quad \text{where } A > \tfrac{1}{2}a\ldots \ldots \ldots \ldots 63$$

11022. (G. Heppel, M.A.)—If A, B, C are the angles of a triangle, transform $(\sin B + \sin C)\sin B \sin C + (\cos C - \cos A)\cos A \sin C + (\cos B - \cos A)\cos A \sin B$ into an expression symmetrical with respect to A, B, C. ..... 61

11041. (Professor Morel.) — Etant donné un triangle rectangle isoscèle OAB, on demande : (1) l'équation générale des hyperboles équilatères circonscrites à ce triangle ; (2) si l'on considère la tangente à l'origine à chacune de ces hyperboles, on sait que cette droite est fixe. On suppose que sur cette droite on prenne un point fixe C. Sur ce point on mène la deuxième tangente à l'une quelconque des hyperboles précédentes. Lieu des points de contact de ces deuxièmes tangentes.—Ce lieu est composé d'une droite fixe et d'une conique. Si le point C se péplace sur la tangente fixe commune à toutes les hyperboles équilatères précédentes, on demande le lieu des centres des coniques précédentes. 59

11062. (D. Biddle.)—The ten figures, 0 ... 9, are arranged at random in two rows. Find the probability that $n$ (and only $n$) of the five lower ones are less than the several figures immediately above them. ..... 127

11072. (Rev. T. R. Terry, M.A.)—Simplify the expression

$$(bc+ad)^3(b^3-c^3)(a^3-d^3)+(ca+bd)^3(c^3-a^3)(b^3-d^3) + (ab+cd)^3(a^3-b^3)(c^3-d^3).$$

..... 47

11078. (A. Gordon, B.A.) — Prove that, if $r < n-1$ be a positive integer, then we shall have

$$\frac{\cos r\alpha_1}{\sin(\alpha_1-\alpha_2)\sin(\alpha_1-\alpha_3)\ldots\sin(\alpha_1-\alpha_n)} = 0$$

$$= \Sigma \frac{\sin r\alpha_1}{\sin(\alpha_1-\alpha_2)\sin(\alpha_1-\alpha_3)\ldots\sin(\alpha_1-\alpha_n)},$$

where the $\Sigma$ refers to a cyclic interchange of the suffixes 1, 2 ... $n$, so that the next term is $\cos r\alpha_2 / \sin(\alpha_2-\alpha_3)\sin(\alpha_2-\alpha_4)\ldots\sin(\alpha_2-\alpha_1)$. ..... 45

11080. (R. Chartres.)—Give an elementary proof of Gregory's Series without the use of $(-1)^{\frac{1}{2}}$. ..... 77

11099. (Professor Gay.)—Par quatre points fixes C, D, E, F, pris dans un ordre quelconque sur une ellipse donnée, on fait passer des parallèles qui coupent la courbe en quatre autres points variables C′, D′, E′, F′ ; trouver (1) les lieux des points d'intersection des droites C′D′ et E′F′, D′C′ et E′F′, E′C′ et D′F′; (2) le lieu d'un point P de la droite C′D′ tel que PC′ : PD′ = K, K étant constant ; et (3) l'enveloppe de cette droite. ..... 127

Digitized by Google

11113. (G. F. Howse.)—Three circles touch one another two and two; through each point of contact another circle is drawn touching the pair at that point and cutting the third at right angles. Show that these three circles are coaxal. ..... 60

11156. (The late Professor Wolstenholme, Sc.D.)—Prove that, if $r$ be any positive integer, $n > 1$, and $\theta \equiv \pi/n$, then we have

$$\int_0^\infty \frac{(\log x)^r}{1+x^n}dx = \theta^{r+1}\frac{d^r}{d\theta}(\operatorname{cosec}\theta), \quad \int_0^\infty \frac{(\log x)^r}{x^n-1}dx = \theta^{r+1}\frac{d^r}{d\theta^r}(\cot\theta). \quad 39$$

11159. (Professor Neuberg)—Sommer la série

$$\frac{x}{1}-\frac{x^2}{2}+\frac{x^5}{5}-\frac{x^6}{6}+\frac{x^9}{9}-\frac{x^{10}}{10}+\frac{x^{13}}{13}-\ldots. \quad \ldots\ldots\ldots 39$$

11167. (Professor Mannheim.)—Un quadrilatère articulé ABCD a son sommet A qui est fixe; les côtés AB, AD tournent autour de A d'angles égaux en sens inverses. On demande le lieu du point C. ... 36

11171. (Professor Niewenglowski.)—Par un point P donné dans le plan d'un parallélogramme ABCD, mener, avec la règle seule, une parallèle à une droite $mn$ située dans ce plan. ..... 88

11172. (Professor Brouilhet.)—Parmi tous les cônes de révolution de même volume, déterminer celui dont la surface latérale est minima. 50

11183. (R. Tucker, M.A.)—P, Q, R are three cotangential points on the curve $ay^2 = x^3$, and PQ, QR, RP meet the curve again in $r$, $p$, $q$. Prove that $\Delta pqr = 2\Delta PQR$; and find the equation to the circle PQR. 78

11184. (The late W. J. C. Sharp, M.A.)—Prove that

$$\begin{vmatrix} 1, & 1, & 1, & \text{to } n \text{ columns} \\ 1, & 0, & 1, & \ldots \quad \ldots \quad \ldots \\ 1, & 1, & 0, & \ldots \quad \ldots \quad \ldots \\ \ldots & \ldots & \ldots & \ldots \quad \ldots \quad \ldots \end{vmatrix} = (-1)^{n-1}. \quad \ldots\ldots\ldots 75$$

11186. (H. MacColl, B.A.)—From the conditional statement, "If all Z is Y, all X is Y," may we logically infer that "all X is Z"? ..... 38

11189. (Rev. T. R. Terry, M.A.)—If

$$N = (x-a)^5(b-c)^5 + (x-b)^5(c-a)^5 + (x-c)^5(a-b)^5,$$

and

$$D = (x-a)^3(b-c)^3 + (x-b)^3(c-a)^3 + (x-c)^3(a-b)^3,$$

where $a$, $b$, $c$ are the roots of the equation $y^3 + 3Py^2 + 3Qy + R = 0$; show that

$$\frac{N}{15D} = (P^2-Q)x^2 + (PQ-R)x + (Q^2-PR). \quad \ldots\ldots\ldots 91$$

11190. (H. J. Jacob.)—On April 1st a man borrows £57, and agreed to repay the lender £60 by ten equal instalments on the 1st of each month following; find what rate of interest per cent. per annum the borrower is paying. ..... 29

11191. (Rev. T. Roach, M.A.)—In an ellipse PQ is drawn perpendicular to SP, and $PS_1$, PQ meet CD in E, Q; prove that the radius of the circle about PEQ varies inversely as FE. ..... 63



Digitized by Google

11194. (R. Chartres.)—Give a simple proof, without infinitesimal changes, that when four straight lines are given, the area enclosed will be a maximum when the figure is concyclic. ..... ..... 35

11201. (G. S. Carr, M.A.)—If $V = f(x, y, p, q, r, \&c.)$, where $y$ is a function of $x$, and $p, q, r, \ldots$ are the successive differential coefficients of $y$; show that the $n^{th}$ variation of V, due to a change in the *form* of the function $y$, is $\delta^n V = \left(\delta y \frac{d}{dy} + \delta p \frac{d}{dp} + \delta q \frac{d}{dq} + \&c.\right)^n V$, where $\delta y$, $\delta p$, &c. follow the law of involution merely, but $\frac{d}{dy}$, &c. the operative law, V being the subject. ..... ..... 80

11203. (Professor B. O. Pierce, M.A.)—PP′ is an arc of continuous curvature. The tangents drawn to the arc at its extremities cut the tangent which is parallel to the chord PP′, at S and S′ respectively. Show that, as PP′ is made to approach zero, the ratio of SS′ to PP′ approaches the limit $\frac{1}{2}$. ..... ..... 31

11210. (Professor Malilal Mallik, M.A.)—Draw (1) through a given point between two given right lines, a right line such that it would be one of the points of trisection of the line; and find (2) how many such lines can be drawn through the point. ..... ..... 29

11213. (Professor Cavallin.)—A quadric is described to pass through two non-intersecting given lines and to touch a given plane. Show that the locus of the point of contact with the plane is a straight line. ... 112

11214. (Professor Chakrivarti, M.A.)—A straight cylindrical rod, of length $l$, symmetrical about its centre line, and of constant flexural rigidity B about any axis perpendicular to its central line, is acted on by a thrust R and opposing couples G in a plane perpendicular to R. Show that the greatest value of $l$ for which a straight central line is a stable configuration for the rod is given by $\frac{\pi^2}{l^2} = \frac{G^2}{4B^2} + \frac{R}{B}$. ..... 111

11221. (Professor Catalan.)—Intégrer l'équation

$$y' = (xy - y^2)/(1 - x^2). \quad \ldots\ldots\ldots\ldots \quad 34$$

11227. (Professor Mannheim.)—On donne un triangle ABC. Du sommet A, on mène la médiane AM qui aboutit au milieu M de BC, et la bissectrice AL qui rencontre la base BC au point L. Du point L, on élève une perpendiculaire à la bissectrice AL; cette droite coupe la médiane au point P et le côté AB au point Q. Démontrer que la perpendiculaire abaissée de P sur BC, et la perpendiculaire élevée de Q à AB, se coupent sur la bissectrice AL. ..... 79

11229. (Professor Neuberg.)—Par un point quelconque P situé à l'intérieur du triangle donné ABC, on mène une parallèle à BC, coupant CA en C′ et AB en B″; une parallèle à CA, coupant AB en A′ et BC en C″; une parallèle à AB, coupant BC en B′ et CA en A″. La probabilité que les segments B′C″, C′A″, A′B″ soient les côtés d'un triangle, est

$$p = 2abc/\{(b+c)(c+a)(a+b)\}. \quad \ldots\ldots \quad 93$$

Digitized by Google

11232. (The Editor.)—Prove that if, in a triangle ABC, the angles be such that $\Sigma(\cos A) = \sqrt{2}$, the circumcentre of the triangle will be on the circumference of the incircle. ..... 63

11238. (H. F. Davis, M.A.)—From the multiplication of $A+x$ by $B+y$, deduce a simple rule for calculating such values as $21\frac{7}{8}$ yards at $19\frac{3}{4}$ pence each, or $16\frac{3}{8}$ yards at $2s.\ 3\frac{1}{2}d.$ each..... 40

11241. (J. J. Barniville.)—Evaluate or seriate

$$\sqrt{1+\sqrt{1+\sqrt{1+\&c.}}}, \quad \log 1+\log\overline{1+\log\overline{\&c.}}\ldots\ldots (1, 2).$$

..... 54

11247. (H. Briggs, M.A.)—Given AC = 1000 links, $\angle$ ACD = 20°, $\angle$ DCB = 35°, find the angle CAB, so as to cut off an area CDB = 2 acres. ..... 35

11251. (Professor de Longchamps.)—Soit P une parabole donnée de foyer F. D'un point M, arbitrairement choisi dans son plan, on mène à cette courbe les normales MA, MB, MC. Les tangentes aux points A, B, C forment un triangle A'B'C'. Démontrer que la circonférence A'B'C' est vue du point M sous un angle constant $\alpha$ tel que $\sin\frac{1}{2}\alpha = \frac{1}{3}$. ..... 34

11253. (Professor Neuberg.)—Soient $u_1$, $u_2$, $u_3$, $u_4$ quatre droites situées dans un même plan. Dans les triangles $u_1u_2u_3$, $u_1u_3u_4$, $u_1u_4u_2$ on mène les médianes qui aboutissent aux milieux des côtés situés sur $u_1$. ..... 32

11254. (Professor Tissot.)—Une circonférence ayant pour centre un foyer d'une conique, si l'on mène, à chaque normale de cette dernière courbe, une parallèle par le point où le rayon vecteur du pied de la normale rencontre la circonférence, les droites ainsi obtenues seront concourantes. ..... 71

11255. (Professor Malet, F.R.S.)—Prove the following method of finding the equation of squared differences of the roots of

$$f(x) \equiv a_0x^n + a_1x^{n-1} + \&c. = 0.$$

Let $$f(x+\tfrac{1}{2}\sqrt{z}) \equiv \phi(x, z) + \sqrt{z}\,\psi(x, z),$$

where $\phi$ and $\psi$ are rational integral functions of $x$ and $z$; then, if we eliminate $x$ between $\phi = 0$ and $\psi = 0$, the resulting equation in $z$ is the required equation. Apply this method to the cubic. ..... 41

11259. (The late Professor Clifford, F.R.S.)—If a hexagon can be inscribed in a cubic curve so that the first, third, fifth sides pass through a fixed point $a$ on the curve, and the second, fourth, sixth pass through a fixed point $b$ on the curve, prove that, when the point $a$ is given, there are eight positions of $b$, which can be found by linear construction when the points of inflexion are known. ..... 49

11260. (Professor Catalan.)—On donne une droite AB, une circonférence C, et une droite CDP, sur laquelle on prend deux points fixes, D et P. Par le point P, on mène, arbitrairement, une transversale qui coupe en E la droite AB, en F et G la circonférence C. On joint E à D, F et G à C, par les droites DE, CF, CG, qui se coupent en M, M'. Le lieu des points M, M' est une conique, dont on demande la discussion. 79

Digitized by Google

11269. (Professor Nilkantha Sarkar, M.A.)—O is the orthocentre of a triangle ABC, and O′ is the centre of its circumscribing circle; A′, B′, C′ are the centres of the circles circumscribing the triangles BOC, COA, AOB respectively. Show that O′ is the orthocentre of A′B′C′, O the centre of its circumscribing circle, A, B, C those of the circles circumscribing B′O′C′, C′O′A′, A′O′B′ respectively, and that all the eight triangles above mentioned have the same nine-point circle. ..... 82

11274. (G. E. Crawford, M.A.) — If we consider the earth as a perfect sphere irregularly dotted with houses, some one house, say B, will be the farthest of any from a given house A, and some other house, say C, will be the farthest from B. Do A and C necessarily coincide? If not, what is the greatest distance between them? [Distances to be measured along the surface.] ..... 78

11283. (H. W. Pyddoke, B.A.) — If $a$ and $b$ are whole numbers taken at random, show that the chance that $a/b$ is in its lowest terms is $6/\pi^2$. ..... 57

11284. (E. M. Langley, M.A.)—If B, C are points of trisection of a straight line ABCD, and AEB, AEGC, AHD, BKLD semicircles on AB, AC, AD, BD, show geometrically that, if circles EFK, GHL be drawn to touch them, they are equal and have a common tangent perpendicular to AD. ..... 67

11285. (A. J. Pressland, M.A.)—Prove that the median triangle of any self-conjugate to a parabola is circumscribed to the parabola. ... 76

11297. (Professor Chakrivarti, M.A.)—The extremities of a uniform heavy beam, of length $2a$, slide on a smooth wire in the form of the curve $r = a(1-\cos\theta)$, the prime radius being vertical, and the vertex of the curve downwards. Prove that, if the beam be placed in a vertical position, and displaced with a velocity just sufficient to bring it into a horizontal position, and $\theta$ be the angle through which the rod has turned after a time $t$,

$$\tan\theta = \tfrac{1}{2}\left\{\epsilon^{(3g/2a)^{\frac{1}{2}}t} - \epsilon^{-(3g/2a)^{\frac{1}{2}}t}\right\}. \text{ ..... } 51$$

11299. (Professor B. O. Pierce, M.A.)—P is any point of a plane closed oval, QPQ′ a straight line drawn in the plane in a given constant direction so that QP, which is equal to PQ′, is a given finite continuous and single-valued function of the coordinates of P. Show that, as P moves around the curve, Q and Q′ will trace out two closed loops the sum of the areas of which is double the area of the given oval. In general, if $QP = \lambda PQ'$, $\lambda+1$ times the area of the oval is equal to the area traced out by Q plus $\lambda$ times the area traced out by Q′. [*See* Wolstenholme, Prob. 2400.] ... 53

11301. (Professor Crofton, F.R.S.)—Prove that the $(r)^{\text{th}}$ differential coefficient of $\exp(e^x)$ is

$$\mathrm{D}^r \exp(e^x) = \exp(e^x)\left\{e^x + \frac{\Delta^2 0^r}{2!}e^{2x} + \frac{\Delta^3 0^r}{3!}e^{3x} + \ldots e^{rx}\right\}. \text{ ..... } 109$$

11303. (Professor de Bozóby.)—Combien de solutions distinctes admet l'équation $x+y+z = n+2$, les indéterminées $x$, $y$, $z$ ayant des valeurs égales ou inégales choisies parmi les nombres 1, 2, 3, ... $n$? ..... 95

Digitized by Google

11304. (The Editor.)—On the prolongation of a chord AB of a circle, find a point P such that, if AC, BD be perpendiculars on the tangent from P, the rectangle AC . BD may be given..... ..... 50

11311. (Mme Vve F. Prime.)—Dans un quadrilatère ABCD, dont les angles B, D sont droits, on joint le sommet C à un point quelconque E de la diagonale BD, et l'on élève en E une perpendiculaire à CE qui rencontre en F, G les côtés AB, AD. Trouver le lieu du point K qui divise GF de telle sorte que GK : KF = BE : ED. ..... 28

11316. (G. E. Crawford, M.A.)—If A, B be two fixed points, and P a movable one such that one bisector of the angle APB passes through a fixed point, find (1) the locus of P; and (2) the envelope of the other bisector. ..... 52

11319. (R. Tucker, M.A.)—D, E, F are points of ex-contact on the sides BC, CA, AB of a triangle. Find the co-ordinates of P, the point of intersection of perpendiculars to the sides at D, E, F. If I be the in-centre, find the equation to the circle on PI as diameter, and derive an expression for $\Sigma\left[a\left\{(1-\cos A)^2-(\cos B-\cos C)^2\right\}\right]$. ..... 76

11320. (D. Biddle.)—Prove the necessary correctness of the following method of filling a *magic square* (the number of whose spaces is odd), so that each row, column, and diagonal shall add up to the same amount. It is not requisite to begin with 1, but the integers must be consecutive, or, at least, have uniform intervals between them. Put the one with which you start in the middle space of the top row, and wherever possible proceed diagonally upwards to the right. The following exceptions occur: (*a*) when it leads above the square, drop to the lowest space of the next column; (*b*) when it leads to the right of the square, go to the extreme left of the row above; (*c*) when obstructed by a previous figure, drop immediately below the last; (*d*) similarly when at right top corner. 25

| | | |
|---|---|---|
| 9 | 2 | 7 |
| 4 | 6 | 8 |
| 5 | 10 | 3 |

11328. (H. Fortey.)—Find the equation to the evolute of the lemniscata of Bernoulli. [This problem was also solved by Mr. Tucker 30 years ago, as Question 1407, in Vol. I. of our series (1st ed., p. 58, 2nd ed., p. 109).] ..... 74

11330. (Professor Ramaswami Aiyar, M.A.)—If similar figures be described on the faces of an isosceles tetrahedron, prove that (1) any four corresponding points or places determine an isosceles tetrahedron; and (2) any four corresponding lines are generators of the same system of a hyperboloid. ..... 81

11334. (Professor Lampe, LL.D.)—Let P be a point of the transverse axis of a conic section whose focal axis is horizontal, T the point of contact of one of the tangents drawn from P to the curve. A material point glides without friction from P to T, the initial velocity being zero. If P be determined in such a way that the time of descent is a minimum, P proves to be a fixed point for a system of confocal conics, and the problem is easily reduced to a well known movement on the chords of a circle. 55

Digitized by Google

11337. (Professor Curtis, M.A., S.J.) — At any point on a curve the radius of the circumcircle of the triangle formed by three tangents, when they become coincident, is $= \frac{1}{4}$ of the radius of curvature at the same point..... 58

11340. (Professor de Longchamps.) — On considère une parabole P dont le sommet est S. Soit AA′ une corde principale de P. Il existe une circonférence Δ tangente, aux points A, A′, aux droites SA, SA′; elle coupe P en deux points B, B′, différents des points A, A′. Démontrer que (1) la distance des droites AA′, BB′ est égale au paramètre $2p$; (2) ayant tracé une droite parallèle à l'axe et touchant Δ en I, le lieu de ce point, quand AA′ est mobile, est la parabole P; et (3) trouver l'enveloppe des circonférences telles que Δ. ..... 93

11343. (R. Tucker, M.A.)—ABC is a circle. ABC, A′B′C′ are two diametral triangles in the circle. Any point (P) on the circumference is joined to the six points; so that PA′, PB′, PC′ cut BC, CA, AB in $d, e, f$; and PA, PB, PC cut B′C′, C′A′, A′B′ in $d', e', f'$. Prove that $def$, $d'e'f'$ coincide with two diameters of the circle. ..... 84

11349. (A. S. Ramsay, B.A.)—If $\rho$, $\rho'$, and $2n$ are respectively the principal radii of curvature and the length of the normal chord at a point on the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, the length of the central perpendicular on the tangent plane at which is $p$; prove that (1)

$$\frac{1}{\rho} + \frac{1}{\rho'} + \frac{1}{n} = p\left\{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right\},$$

and (2), if $r$ is the radius vector of the point,

$$n = a^2b^2c^2 / \left[p\left\{p^2(r^2 - a^2 - b^2 - c^2) + b^2c^2 + c^2a^2 + a^2b^2\right\}\right]. \quad \ldots\ldots \quad 94$$

11351. (G. Heppel, M.A.)—If $\alpha$, $\beta$, $\gamma$, $\delta$ are the interior angles made by the diagonal AC of a quadrilateral with the sides AB, BC, CD, DA, and $r$ and R the radii of the circles passing through the intersections of the internal and external bisectors of the angles of the quadrilateral, prove that

$$r = \tfrac{1}{4}\mathrm{AC}\,\frac{\cos\frac{1}{2}(\alpha+\gamma)\cos\frac{1}{2}(\beta-\delta) \sim \cos\frac{1}{2}(\beta+\delta)\cos\frac{1}{2}(\alpha-\gamma)}{\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\gamma+\delta)\cos\frac{1}{2}(\alpha-\gamma)\cos\frac{1}{2}(\beta-\delta)},$$

$$\mathrm{R} = \tfrac{1}{4}\mathrm{AC}\,\frac{\cos\frac{1}{2}(\alpha+\gamma)\cos\frac{1}{2}(\beta-\delta) + \cos\frac{1}{2}(\beta+\delta)\cos\frac{1}{2}(\alpha-\gamma)}{\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\gamma+\delta)\cos\frac{1}{2}(\alpha-\gamma)\cos\frac{1}{2}(\beta-\delta)}. \quad 51$$

11354. (W. J. Greenstreet, M.A.)—A, B, C, D are the points of intersection of a circle with an equilateral hyperbola. If AB passes through the centre of the hyperbola, then CD will pass through the centre of the circle. ..... 53

11355. (A. J. Pressland.)—If DEF be a transversal of the triangle ABC, cutting BC in D, CA in E and AB in F, prove that

$$\frac{\mathrm{AE}\,.\,\mathrm{AF}}{\mathrm{CA}\,.\,\mathrm{AB}} + \frac{\mathrm{AF}\,.\,\mathrm{BD}}{\mathrm{AB}\,.\,\mathrm{BC}} + \frac{\mathrm{CD}\,.\,\mathrm{AE}}{\mathrm{BC}\,.\,\mathrm{CA}} = 0. \quad \ldots\ldots \quad 72$$

Digitized by Google

11362. (W. W. Johnson.)—Given three points and three straight lines in a plane, the determinant of the nine perpendiculars from the points to the lines is equal to twice the product of the areas of the triangles formed by the points and by the lines, divided by the radius of the circle circumscribing the latter. ..... 90

11363. (William Hoover.) — A man runs a race, starting with velocity $v$, the direction of the wind appearing to him to make an angle $\alpha$ with his course. It then appears uniformly to veer round through an angle $\alpha$ during the race. The wind blows uniformly throughout and at right angles to his course. Show that, if $t$ be the time of the race, the length of the course is $vt\frac{\tan\alpha}{\alpha}\log(2\cos\alpha)$. ..... 68

11366. (Hilda Hudson.)—If a line AB is bisected in C and divided unequally in D, either internally or externally, to prove that the sum of the squares on AD, DB is equal to twice the sum of the squares on AC, CD. ..... 36

11368. (Professor Haughton, F.R.S.) — Two rectangular simple vibrations, one of which has double the periodic time of the other, act upon the same particle; find the path described by the particle. ..... 57

11371. (Professor Minchin, M.A.) — The axes of three coplanar screws, of pitches $p_\alpha$, $p_\beta$, $p_\gamma$, form a triangle whose sides are $a$, $b$, $c$, respectively; prove that the pitches of the two screws (other than that perpendicular to their plane) that can be drawn reciprocal to them through any point O in their plane, are the roots of the equation

$$\frac{ap}{x+p_\alpha}+\frac{bq}{x+p_\beta}+\frac{cr}{x+p_\gamma}=0,$$

where $p$, $q$, $r$ are the perpendiculars from O on $a$, $b$, $c$ (reckoned all positive when O is inside the triangle). ..... 127

11378. (Professor de Longchamps.)—Démontrer que

$$\tfrac{1}{2}\Sigma a^2(b-c)^2(ab+ac-2bc)^2=\left\{l^2c^2+c^2a^2+a^2b^2-abc(a+b+c)\right\}^2. \quad 112$$

11384. (Professor Nixon, M.A.)—If A is the intersection of a direct and a transverse common tangent to two circles, and AL perpendicular to their line of centres, show that L is a limiting point of the circles. Deduce *geometrically* that, if D is the diameter of the circle cutting the three ex-circles of a triangle orthogonally, $r$ the in-radius and $s$ the semi-perimeter of the triangle, then $D^2=r^2+s^2$. [A longer proof than here indicated of this theorem, which forms Question 8343, is given by Mr. Davis in Vol. XLVIII. of our *Mathematics*.] ..... 87

11386. (Professor Genese, M.A.)—The bisectors of the angles between opposite sides produced of a quadrilateral in a circle meet on the join of the mid-points of the diagonals. ..... 84

11387. (The Editor.)—If a parabola move parallel to itself in such wise that its vertex is always on a fixed line, prove that the locus of the pole of a fixed straight line with respect to the parabola is a straight line. ..... 86

Digitized by Google

11388. (The Bachelors of Cambridge.)—

In ninety-two and ninety-three
Twelve balls of ours we fix to be:
Ten Girton maids, Urania's pets,
We bid, in unnamed quintuplets,
To each to come,
Five left at home,
While dance a dozen different sets.
The beauties don't deny, but all
Cry: "Most unmathematical!
Shall we, for random invitations,
Descend to work at Combinations?
Nay—let the males, to learn themselves,
Pile divers fives of us, in twelves
Of fives, till found by skill or chance is
Each separate twelve, that gives the dances
To every name
In tale the same."
The saucy Ten's demand,
That they may fit their fancies
With all the dozens of fives in hand—
We are bound to find an answer to it;
But, O ye Muses, how to do it?
..... 82

11388A. (Let down to his own inches by Non-Œdipus.)—

In ninety-two and ninety-three
Twelve balls of ours are fixed to be:
Ten Girton maids, Urania's pets,
We bid, in unnamed quintuplets,
To each to come,
Five left at home
(*The alternate sets*).
The beauties don't deny, &c.
[The rest as above.]
..... 97

11391. (R. Lachlan, M.A.)—A conic is inscribed in a quadrilateral, and three other conics are drawn to osculate the given conic at the same point and pass respectively through the extremities of the diagonals of the quadrilateral. Show that the poles with respect to these conics of the three diagonals are coincident; and that the three conics and the given conic have another common tangent. ..... 86

11392. (E. M. Langley, M.A.)—SY is drawn perpendicular to the tangent at P to a conic from the focus S. The line joining P to the vertex A meets the corresponding directrix in E. Show that AY is a median of the triangle SYE. ..... 128

11408. (Professor Genese, M.A.)—Given a cycle C (*i.e.*, a circle with a definite sense of description), and a straight line AB. Prove that the bisector of the angle which any tangent to C makes with AB touches a parabola; and that the bisector of the angle the tangent makes with BA touches a different parabola; further, the parabolæ and the cycle have AB as common chord. ..... 102

11411. (Professor Neuberg.)—On donne une suite d'hyperboles ayant les mêmes asymptotes OX, OY. On prend sur chacune de ces courbes un point M tel que le secteur compris entre la courbe, la droite OM, et une droite fixe menée par O, ait une aire constante. Trouver le lieu du point M. ..... 86

11418. (Professor Liègor.)—On considère un parallélogramme ABCD et un point extérieur O, compris entre les prolongements des côtés AD, BC. On joint le point O aux sommets A, B, C, D; soient E, F, G les points de rencontre respectifs des couples de droites OA et DC, OB et DC, OB et AC. Démontrer que le quadrilatère AEFG équivaut à la somme des trois triangles BCG, CFO, DEO. ..... 119

11420. (The Editor.)—Inscribe in a circle a quadrilateral ABCD, having the diagonal AC given in direction, the diagonal BD in magnitude, and the point E in which the sides AB, CD meet. ..... 98

Digitized by Google

11423. (R. W. D. Christie.)—Prove that the principle of "Casting out the Nines" may be extended to any number ending in 1, 3, 7, or 9.

[*Ex. gr.*: To test 973112691 for 37, we have $1+2+3=6$; $9+1+7=17$; $6+1+9=16$; $17-16=1$ (ten) and $6-16=-10$; therefore $10-10=0$: and the given number = 37M.] ..... 111

11426. (Col. H. Nuttall.) — In the *Monist Quarterly Magazine* for January, 1891, in an article by Charles S. Pierce, entitled "The Architecture of Theories," he states that "within another century our grandchildren will surely know whether the three angles of a triangle are greater or less than 180°; that they are exactly that amount is what nobody ever can be justified in concluding. It is true that according to the axioms of geometry the sum of the three angles of a triangle is precisely 180°; but these axioms are now exploded, and geometers confess that they, as geometers, know not the slightest reason for supposing them to be precisely true." Required to know whether it is to be understood that the axioms of geometry *are* exploded as here stated ..... 126

11429. (E. M. Langley, M.A.) — Two circles are described, each passing through one end of the third diagonal of a cyclic quadrilateral and the middle points of the other two. Show that the third diagonal is a common tangent. ..... 116

11431. (Rev. G. Richardson.) — Prove that the two points where the tangent to a hyperbola meets the asymptotes are concyclic with the two foci. ..... 108

11437. (Professor Sylvester.) — A, B, C ... K, L, M are integers: the ratios B/A, C/B, D/C ... diminish continually towards unity as their limit. Prove that if $B-A < A^{\frac{1}{2}}$ the series must be an infinite arithmetical progression, or will become such on lopping off a sufficient number of the terms at the beginning. ..... 97

11440 (Professor Morley, M.A.) — Given a triangle *abc*, show that there are points *x*, *y* such that the triangles *axy*, *ybx*, *xyc*, *abc* are similar; and construct the points. ..... 100

11444. (Professor Zerr.) — Prove that the intercept on the axis of abscissae made by the tangent and normal at any point on an ellipse varies as $\rho^{\frac{3}{2}}/x$, where $\rho$ is the radius of curvature and $x$ the abscissa of the point. ..... 109

11445. (Professor Davis.)—If the centres of three equilateral triangles described on the three sides of a scalene triangle be joined by straight lines, prove by geometry that the triangle thus formed is equilateral. ..... 118

11446. (Professor Gob.)—Soient O le centre, et F, F′ les foyers d'une ellipse. On prolonge un rayon quelconque OM de MM′ = OM, et l'on mène les tangentes M′P, M′Q. P, Q étant les points de contact, démontrer la relation $4M'P \cdot M'Q = 3M'F \cdot M'F'$ ..... 103



Digitized by Google

11447. (Professor Laisant.)—Deux ellipses égales et de même centre O ont leurs grands axes perpendiculaires ; P est un point quelconque de la première, Q et Q′ les points de la seconde où les tangentes sont perpendiculaires à OP, et M, M′ les points milieux de PQ, PQ′ ; $a$ et $b$ étant les demi-axes des deux ellipses, démontrer que OMPM′ est un parallélogramme dont les côtés sont $\frac{1}{2}(a+b)$ et $\frac{1}{2}(a-b)$, et dont les bissectrices des angles ont les directions des axes de deux courbes. ..... 106

11449. (Professor Tissot.)—Le centre d'un cercle bi-tangent à une conique, la projection d'un point quelconque de la conique sur la corde des contacts et l'intersection de la normale en ce point avec l'axe qui ne contient pas le centre du cercle, sont en ligne droite..... 117

11452. (Professor de Longchamps.)—Résoudre les équations

$$(y-z+b^2-c^2)^2 = 4a^2x, \quad (z-x+c^2-a^2)^2 = 4b^2y, \quad (x-y+a^2-b^2)^2 = 4c^2z.$$

..... 128

11453. (Professor Lampe, LL.D.)—The arc BC, between two points B and C of the same parallel (latitude $\phi$), has the length of $k$ kilometres. Join B and C by the arc (BC) of the great circle through B and C, and calculate the difference BC−(BC), which is moreover the distance of the centres of BC and (BC). Put $k = 1$ kilometre, the radius of the earth being $2 \cdot 10^4/\pi$ kilometres. ..... 114

11454. (Professor Neuberg.)—Soient P, Q deux points inverses par rapport au cercle circonscrit au triangle ABC ; $P_a$, $P_b$, $P_c$ les projections de P sur les côtés BC, CA, AB ; $Q_a$, $Q_b$, $Q_c$ celles de Q. La droite PQ rencontre la circonférence ABC aux points R, S. Démontrer que les côtés correspondants des triangles $P_aP_bP_c$, $Q_aQ_bQ_c$ font des angles égaux, en sens contraire, avec la droite de Simson du point R. ..... 98

11455. (Professor Mannheim.)—On donne une circonférence de cercle et les tangentes $sa$, $sb$ à cette courbe. Du point $s$, on mène une droite arbitraire qui rencontre la circonférence en $p$ et en $q$ ; démontrer que les distances des points de contact $a$, $b$, aux points $p$, $q$ sont proportionelles, c'est-à-dire que $\frac{ap}{aq} = \frac{bp}{bq}$. ..... ..... 108

11460. (W. S. B. Woolhouse, F.R.A.S.)—Sets of Magic Squares, of specified dimensions, may be constructed and magically associated in the compartments of a square of larger dimensions such that the numbers contained in every square of the specified dimensions that can be taken therefrom shall constitute a perfect Magic Square. Show how to form Magic Squares magically associated. ..... 104

11461. (Col. de Land.)—A engaged in business which paid him a profit at the rate of 40 per cent. per annum. At the end of three months he added his profits to his business, and took in B as a partner, and then found their profits to be at the rate of 60 per cent. per annum. At the end of four months after B entered the firm they added their profits to the business, and found that A's capital was to B's as 11 to 7. They then took in C as a partner and found their losses to be at the rate of 20 per cent. per annum. At the end of six months after C entered the firm the losses were adjusted, and it was found that B's capital was to C's as

Digitized by Google

7 to 4. D now proposed to enter the firm, reorganize it, leave the capital stock the same, each partner to own an equal share, D to put in $5940, and A, B, and C to each put in or take out capital until the shares were equalized. Required (1) the capital each of the partners began with, and (2) the amount of capital each must put in or take out to equalize the shares. ... . ..... 102

11463. (Artemas Martin, LL.D.) — A farmer commenced sheep-raising with a ewe lamb which had twins when one year old, one being a ewe. All the female progeny produce in this way. He sells all the sheep to the butcher when they are 4 years old, and the ewes have no lambs the last year of their lives. Thus each ewe produces three pairs of twins, one of each of which is a ewe. If none die, what will be the number in the flock at the end of 10 years, and how many will be sold the last year ?..... 107

11466. (R. Tucker, M.A.)—Parallelograms $\Omega A\Omega'D$, $\Omega B\Omega'E$, $\Omega C\Omega'F$ are formed, where $\Omega$, $\Omega'$, are the B.-points of a triangle. Prove that D, E, F lie on a circle (equal the circumcircle) which cuts the sides of ABC in points $(p, q, r)$, $(p', q', r')$, so as to make the angles

$$\Omega pA = \Omega' p'A = \omega = \&c. \quad \ldots\ldots 125$$

11470. (E. M. Langley, M.A.)—In a given triangle ABC an inversely similar triangle is inscribed with its vertices P, Q, R on BC, CA, AB respectively. If P = A, show that the median of PQR through P passes through a fixed point on the symmedian of ABC through A. ..... 103

11471. (Rev. G. Richardson.)—ABC is a triangle, A' the mid-point of BC, and P the pole of BC, with regard to the circumcircle; through the mid-point of A'P a line is drawn parallel to BC, meeting AB, AC produced in Q, R; then A, P, Q, R are cyclic. ..... 104

11475. (D. Biddle.)—In an airless region, a marble falls from rest on to an inclined plane of indefinite extent. The elasticity of the marble and of the plane being perfect, and the force of gravity $g$ being regarded throughout as practically identical, prove that ($t$ being the time occupied before the first impact) recurring impacts will take place at intervals of $2t$, and at points which are the projections, on the plane, of the positions which would be reached at the respective intervals if the marble were falling free; progress, in fact, being made at the same rate as if the marble rolled down the plane from the projection on it of the original point of rest. ..... 99

11479. (Morgan Brierley.)—If perpendiculars AD, BE, CF, from the vertices to the sides of the triangle ABC, intersect in P; prove that

$$\frac{AB \cdot BC}{AP \cdot CP} = \frac{BE}{PE'}, \quad \frac{AC \cdot BC}{AP \cdot BP} = \frac{CF}{PF'}, \quad \frac{AC \cdot AB}{CP \cdot BP} = \frac{AD}{PD}. \quad \ldots\ldots 119$$

11480. (Professor Sylvester.)—Let H $(x/a)$ mean $x/a$ when the fractional part of $x/a$ is $\frac{1}{2}$, and in the contrary case the nearest integer to $x/a$. Let the successive prime numbers 2, 3, 5, 7, 11, ... be denoted by $p_1$, $p_2$, $p_3$, ... $p_i$. Then, supposing that $p_i$ is $(2n)^{\frac{1}{2}}$ or the prime number next below it, $n - \Sigma H(n/p_i) + \Sigma H(n/p_i p_j) - \Sigma H(n/p_i p_j p_k) + \ldots$ (or, if it is a

Digitized by Google

fraction, the integer part of it) represents the number of prime numbers greater than $n$, and not greater than $2n$.

Ex. $n = 47$, $p_1, p_2, p_3, \ldots p_i = 2, 3, 5, 7,$

$$47 - H\left(\tfrac{47}{2}\right) - H\left(\tfrac{47}{3}\right) - H\left(\tfrac{47}{5}\right) - H\left(\tfrac{47}{7}\right) + H\left(\tfrac{47}{6}\right) + \left(\tfrac{47}{10}\right) + H\left(\tfrac{47}{14}\right) + H\left(\tfrac{47}{15}\right) + H\left(\tfrac{47}{21}\right) + H\left(\tfrac{47}{35}\right) - H\left(\tfrac{47}{30}\right) - H\left(\tfrac{47}{42}\right) - H\left(\tfrac{47}{70}\right) - H\left(\tfrac{47}{105}\right) + H\left(\tfrac{47}{210}\right)$$

and there are 9 primes between 47 and 94, viz., 53, 59, 61, 67, 71, 73, 79, 83, 89. ..... 113

11490. (Professor Neuberg.)—On donne un angle droit XOY et un point A sur OX. Soit B un point variable sur OY. Sur la droite AB on prend le segment AM = OB. Le lieu de M est une courbe considérée par Barrow, qui a déterminé analytiquement la tangente au point M. Trouver cette tangente par des considérations cinématiques..... 114

11493. (Professor Schoute.)—Of a quadrangle two opposite sides are equal. Prove that the lines bisecting orthogonally the other sides coincide or intersect *without* the quadrangle. ..... 115

11508. (I. Arnold.) — Construct a right-angled triangle, having given the median lines from the extremities of the hypotenuse. ..... 117

11511. (R. W. D. Christie.)—Prove that the sum of four integral squares can in general be resolved into four others; and give a simple rule for the transformation..... ..... 120

## APPENDIX.

Unsolved Questions ..... 129

Digitized by Google

# MATHEMATICS

FROM

*THE EDUCATIONAL TIMES.*

WITH ADDITIONAL PAPERS AND SOLUTIONS.

---

**11320.** (D. Biddle.)—Prove the necessary correctness of the following method of filling a *magic square* (the number of whose spaces is odd), so that each row, column, and diagonal shall add up to the same amount. It is not requisite to begin with 1, but the integers must be consecutive, or, at least, have uniform intervals between them. Put the one with which you start in the middle space of the top row, and wherever possible proceed diagonally upwards to the right. The following exceptions occur: (*a*) when it leads above the square, drop to the lowest space of the next column; (*b*) when it leads to the right of the square, go to the extreme left of the row above; (*c*) when obstructed by a previous figure, drop immediately below the last; (*d*) similarly when at right top corner.

| | | |
|---|---|---|
| 9 | 2 | 7 |
| 4 | 6 | 8 |
| 5 | 10 | 3 |

---

*Solution by* W. S. B. Woolhouse, F.R.A.S.; *and* Rev. Dr. Kolbe.

In what follows, the inquiry is limited to squares of *five*, having twenty-five compartments, which considered alone are somewhat interesting. Suppose the consecutive numbers N = 1 to 25 to be required to be inserted in suitable compartments of the square. Let the number N be generally represented by the formula $N = 5A + B + 1$. Then the values of A or B will consist of the numbers 0 1 2 3 4. In the first horizontal line of the square, suppose the numbers A and B to have the five values $a\ b\ c\ d\ e$ and $a'\ b'\ c'\ d'\ e'$, respectively. And, taking all the five bands of the square, suppose that these values are permuted in the following manner:—

| A ... | | | | | | B ... | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $a$ | $b$ | $c$ | $d$ | $e$ | | $a'$ | $b'$ | $c'$ | $d'$ $e'$ |
| | $b$ | $c$ | $d$ | $e$ | $a$ | | $d'$ | $e'$ | $a'$ | $b'$ $c'$ |
| | $c$ | $d$ | $e$ | $a$ | $b$ | | $b'$ | $c'$ | $d'$ | $e'$ $a'$ |
| | $d$ | $e$ | $a$ | $b$ | $c$ | | $e'$ | $a'$ | $b'$ | $c'$ $d'$ |
| | $e$ | $a$ | $b$ | $c$ | $d$ | | $c'$ | $d'$ | $e'$ | $a'$ $b'$ |

By inspection of the cluster A, it is seen that every horizontal and vertical ine contains identically the five values $a\ b\ c\ d\ e$, only differently placed,



Digitized by Google

and as these consist of the numbers 0 1 2 3 4 permuted, the summation in every case must give 10.

The same remark applies to one of the diagonals; but the other diagonal is exceptional, and contains the symbol $e$ repeated five times. In order that this diagonal may amount to the same sum, it will only be requisite to have $e = 2$. The cluster B fulfils the magic conditions throughout, so that $a'\ b'\ c'\ d'\ e'$ may be the numbers 0 1 2 3 4 taken in any order.

The following is an example :—

| A | B |
|---|---|
| 3 4 0 1 2 | 1 3 0 2 4 |
| 4 0 1 2 3 | 2 4 1 3 0 |
| 0 1 2 3 4 | 3 0 2 4 1 |
| 1 2 3 4 0 | 4 1 3 0 2 |
| 2 3 4 0 1 | 0 2 4 1 3 |

$N = 5A + B + 1$

| | | | | |
|---|---|---|---|---|
| 17 | 24 | 1 | 8 | 15 |
| 23 | 5 | 7 | 14 | 16 |
| 4 | 6 | 13 | 20 | 22 |
| 10 | 12 | 19 | 21 | 3 |
| 11 | 18 | 25 | 2 | 9 |

In this square the numbers 1, 2, 3, &c., proceed consecutively according to the rule stated in the question. But this rule is very limited in its application. It will fail unless the number 1 is started upon one of the five squares now occupied by the numbers 1 to 5.

For a more extensible system of arrangement let the values of A and B be

| A ... | B ... |
|---|---|
| $a\ b\ c\ d\ e$ | $a'\ b'\ c'\ d'\ e'$ |
| $c\ d\ e\ a\ b$ | $d'\ e'\ a'\ b'\ c'$ |
| $e\ a\ b\ c\ d$ | $b'\ c'\ d'\ e'\ a'$ |
| $b\ c\ d\ e\ a$ | $e'\ a'\ b'\ c'\ d'$ |
| $d\ e\ a\ b\ c$ | $c'\ d'\ e'\ a'\ b'$ |
| &c. &c. | &c. &c. |

Here the permutations are more general, and the magic conditions are perfectly fulfilled by *every cluster of five* in any part of these series, and the numbers $a\ b\ c\ d\ e$ and $a'\ b'\ c'\ d'\ e'$ may be quite arbitrary arrangements of the numbers 0 1 2 3 4.

To show how very comprehensive this system is, an example is here given. Take the following values :—

| | | | | | |
|---|---|---|---|---|---|
| A ... 3 4 0 1 2 | 0 1 2 3 4 | 2 3 4 0 1 | 4 0 1 2 3 | 1 2 3 4 0 | &c. |
| B ... 1 0 4 3 2 | 3 2 1 0 4 | 0 4 3 2 1 | 2 1 0 4 3 | 4 3 2 1 0 | &c. |

Here the figures in B are those of particular groups of A reversed, but that is only incidental and is of no consequence. The first *five square* on the left of the following *nine square* is readily filled in by means of the formula

$$N = 5A + B + 1;$$

and afterwards the remainder of the table is filled in by simple repetition after every fifth line and every fifth column.

Digitized by Google

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 17 | 21 | 5 | 9 | 13 | 17 | 21 | 5 | 9 |
| 4 | 8 | 12 | 16 | 25 | 4 | 8 | 12 | 16 |
| 11 | 20 | 24 | 3 | 7 | 11 | 20 | 24 | 3 |
| 23 | 2 | 6 | 15 | 19 | 23 | 2 | 6 | 15 |
| 10 | 14 | 18 | 22 | 1 | 10 | 14 | 18 | 22 |
| 17 | 21 | 5 | 9 | 13 | 17 | 21 | 5 | 9 |
| 4 | 8 | 12 | 16 | 25 | 4 | 8 | 12 | 16 |
| 11 | 20 | 24 | 3 | 7 | 11 | 20 | 24 | 3 |
| 23 | 2 | 6 | 15 | 19 | 23 | 2 | 6 | 15 |

Any square having 25 compartments which can be taken within the range of this compact collection of family compartments will contain the numbers from 1 to 25 so placed as to form a complete magic square. The middle square of the family has the least number (1) at the centre. Twenty-five magic squares are thus exhibited; and one may be easily selected, by inspection, such that any given number may occupy any stated compartment. Any selection might be facilitated by a piece of cardboard with a square piece cut out to exhibit the required square.

By examining the particular track of the consecutive numbers 1, 2, 3, &c., in any one of these *five squares*, we deduce the following rule for filling them in independently:

Go 1 up and 2 to the right; if obstructed, 2 up.

This rule is quite general, and the operator can begin with 1 upon *any compartment*.

It will be understood, in counting upwards, that on reaching the top the counting must be continued from the foot; and, in counting to the right and arriving at the margin of the square, that the counting must be continued from the left-hand margin.

The general rule here propounded admits of being modified into four separate forms, viz.:—

I. Go 1 up and 2 to the right; if obstructed, 2 up.
II. Go 1 up and 2 to the left; if obstructed, 2 up.
III. Go 1 down and 2 to the right; if obstructed, 2 down.
IV. Go 1 down and 2 to the left; if obstructed, 2 down.

Whichever of these is chosen, it should be adhered to throughout. The principal steps are analogous to the knight's move in chess.

[Rev. Dr. Kolbe proves the theorem thus:—Imagine the whole paper to be plotted out into small squares. Begin anywhere with 1, and write the consecutive numbers up to $n$ diagonally upwards ($n$ being odd). Let these numbers 1 to $n$ be repeated in that diagonal as in wall-paper pattern. In the space under $n$ begin $n+1$ diagonally up to $2n$; these $n$

Digitized by Google

numbers being also repeated both ways along that line. And so on. Thus the numbers 1 to $n^2$ will be arranged in a zigzag line running diagonally upwards, the break of the zigzag always occurring at the multiples of $n$. If this pattern be repeated in every direction, and a square of $n^2$ compartments mapped out in such a way that 1 is in the middle of the top row, the rule given in the question will be manifest, with the reasons for the apparent exceptions. It will be found that the top row will go from the middle to the right-hand corner in A.P. ($d = n+2$) from 1 to $\frac{1}{2}(n^2+n)$; and then from the left-hand corner in the same progression from $\frac{1}{2}(n^2+n+4)$ to $n^2-1$. The sum of these two series is $\frac{1}{2}n(n^2+1)$. Further, each horizontal row has $n$ figures, of which $n-1$ are less by 1, and the other one greater by $n-1$, than the corresponding figure of the row above it. Thus each horizontal row adds up to the same amount. The vertical columns can easily be calculated; for each of them (on the same principle) is equal to the middle column, which is a simple A.P. of $n$ terms from 1 to $n^2$. The diagonal from left to right upwards is obviously a set of $n$ consecutive numbers from $\frac{1}{2}(n^2-n+2)$ to $\frac{1}{2}(n^2+n)$; while the other diagonal consists of two A.P.'s with difference $n+3$, which can easily be summed: *e.g.*, the two extremes are $\frac{1}{2}(n^2+n+4)$ and $\frac{1}{2}(n^2-n-2)$, whose average is $\frac{1}{2}(n^2+1)$. Of course any number $r$ may be added to all the figures without interfering with the properties of the *magic square*. A glance at the accompanying wall-paper pattern for a square of three will make the process clearer, the original zigzag line being italicised.]

| | | | | | | |
|---|---|---|---|---|---|---|
| *5* | 7 | 3 | *5* | 7 | 3 | *5* |
| 9 | 2 | 4 | 9 | 2 | 4 | *9* |
| 1 | 6 | 8 | 1 | *6* | *8* | 1 |
| 5 | 7 | *3* | *5* | *7* | 3 | *5* |
| 9 | *2* | *4* | 9 | 2 | 4 | 9 |
| *1* | 6 | 8 | 1 | 6 | 8 | 1 |
| *5* | 7 | 3 | *5* | 7 | 3 | *5* |

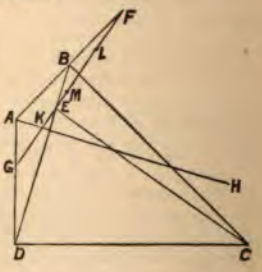
---

**11311.** (Mme Vve F. Prime.)—Dans un quadrilatère ABCD, dont les angles B, D sont droits, on joint le sommet C à un point quelconque E de la diagonale BD, et l'on élève en E une perpendiculaire à CE qui rencontre en F, G les côtés AB, AD. Trouver le lieu du point K qui divise GF de telle sorte que GK : KF = BE : ED.

---

*Solution by* Professor Schoute.

La droite FG enveloppe la parabole (P), dont C est le foyer et BD la tangente au sommet. D'après le théorème que les ponctuelles déterminées par les tangentes à la parabole sur deux d'entre elles sont semblables, le point L symétrique de K par rapport au milieu M de FG parcourt la parabole (P) et M décrit une tangente de (P). Donc K parcourt le diamètre de (P) conjugué à cette tangente, c.-à-d. la droite AH par A perpendiculaire à BD. Ce lieu rectiligne révèle son caractère de parabole dégénérée, en ce que K parcourt deux fois




Digitized by Google

la partie de la droite située en dehors de (P) et ne pénètre pas à l'intérieur de (P).

Vérifions le résultat par la géométrie élémentaire. Soit K le point d'intersection de BD et de la perpendiculaire AH sur BD. Les triangles AKG et DEC sont semblables, les côtés de l'un étant normaux à ceux de l'autre. La même relation existe entre les triangles AKF et BEC. Ces deux couples de triangles semblables donnent

$$GK : CE = AK : DE, \quad KF : EC = KA : EB.$$

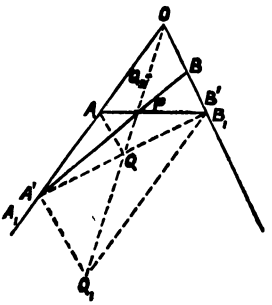
On a donc en effet $GK : KF = BE : ED$.

---

**11210.** (Professor Malilal Mallik, M.A.)—Draw (1) through a given point between two given right lines, a right line such that it would be one of the points of trisection of the line; and find (2) how many such lines can be drawn through the point.

*Solution by* Professor Ratnam Aiyar; J. C. St. Clair; *and others.*

Let OA, OB be the given lines, and P the given point. Join OP and take $PQ = \frac{1}{2}OP$. Draw QA parallel to OB and QB parallel to OA. Then from similar triangles OPA, B′PQ and OPB, A′PQ it is evident that $PB' = 2PA$ and $PA' = 2PB$.




If we take $PQ_1 = 2OP$ we shall, by a similar construction, obtain the same two lines. For in the first case, since A′B, AB′, OQ are the medians of the triangle OA′B′, $OA' = 2OA$. In the second case we have, by similar triangles,

$$OA' : OA = OQ_1 : OQ = 2 : 1.$$

Hence $A' \equiv A_1$. We may obtain two identical pairs of lines trisected *externally* at P by taking, on the opposite side of P, $PQ_2 = \frac{1}{2}PO$, $PQ_3 = 2PO$.

---

**11190.** (H. J. Jacob.)—On April 1st a man borrows £57, and agrees to repay the lender £60 by ten equal instalments on the 1st of each month following; find what rate of interest per cent. per annum the borrower is paying.

*Solution by* Rev. T. R. Terry, M.A.; E. Rutter; *and others.*

First, taking the interest as simple interest, the borrower might be considered as having an equivalent to the use of £57 for $5\frac{1}{2}$ months;

$$\text{therefore rate of interest} = \frac{3 \times 100 \times 12}{57 \times 5\frac{1}{2}} = 11{\cdot}4832536.$$

Digitized by Google

Secondly, supposing the interest due every month, or every instant, then, if R = amount of £1 for one month, we have

$$19R^{10} = 2\{R^9 + R^8 + \ldots + 1\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

If we attempt to solve this equation by HORNER's method, the work is very long. Using logarithms, we have

$$E = \log \frac{R^{10}(R-1)}{R^{10}-1} = \log \frac{2}{19} = \bar{1}\cdot 0222764\,;$$

when R = 1·009, E = $\bar{1}$·0212538; when R = 1·01, E = $\bar{1}$·0235865. These values give R = 1·0094384 as a first approximation to the value of R in (1). On trying 1·009438, 1·009437, 1·009436, we see that a very close approximation is R = 1·00943626. Using this value $R^{12}$ = 1·1193005; therefore rate of interest is 11·93005.

---

**8013.** (Rev. T. P. KIRKMAN, M.A., F.R.S.)—

THE REVENGE OF OLD KING COLE.

"Full oft ye have had your fiddlers' fling,
For your own fun over the wine;
And now," quoth Cole, the merry old king,
"Ye shall have it again for mine.
My realm prepares for a week of joy,
At the coming of age of a princely boy.
Of the grand six days' procession in square,
In all your splendour dressed,
Filling the city with music rare
From fiddles five abreast,

"Ye know; but more—swear all amain
In all your fives, as ye love to do,
'Ye four sots, never I fiddle again,
This week abreast with any of you.'

"Swear it, or not; that law, of my will,
Must rule the array for every day;
And now ye have three months' warning.
The thirty fives, your squares to fill,
Be ready that Monday morning.

"Fail ye—by all that overawes
The soul of fiddling knave,
My barber from your guzzling jaws
Shall ears full fifty shave."

No little sport
Had city and court
With fiddlers in their fives,
And tales in the air
Of their beaten despair
To the last in fight; for it never came right
With the closing Saturday square.

But, when the big moment arrives,
Forth march those twenty-five fiddlers gay,
While betting hundreds book the array,
Some with hopes and some with fears,
To see that gory waste of ears.

More merry and more they parade and play
From morn to eve, till Saturday
Proclaims the monarch's riddle guessed;
For never a pair had twice abreast
Composed the grand display.

And how did they manage to save their ears?
By paying down every rascal a year's
Full wage to the Royal Astrologer—
Solemn old fox, as learned as old:
None in the land so famed for his knowledge, or
Half so greedy of gold.

He did it but once: win nobler fame
By doing it Q times, counting his way,
Thus—keeping the Monday fives ever the same,
Q times you add five squares, one for each day,
Q times keeping whole the rule of King Cole.

Pray let me ask you for this maximum Q,
And for 5Q and one squares, such that twain
Shall be none, that a common five contain;
For surely you, that went to college, are
More than a match for the old Astrologer,

Do you lack a key? I beg you to see
Cam. and Dub. Math. J., N.S., Vol. V.,
Page 256 of a note by me.

Digitized by Google

*Solution by* H. J. WOODALL.

A solution would be

(1)

| | | | | |
|---|---|---|---|---|
| A | B | C | D | E |
| F | G | H | I | J |
| K | L | M | N | O |
| P | Q | R | S | T |
| U | V | W | X | Y |

(2)

| | | | | |
|---|---|---|---|---|
| A | G | M | S | Y |
| F | L | R | X | E |
| K | Q | W | D | J |
| P | V | C | I | O |
| U | | H | N | T |

(3)

| | | | | |
|---|---|---|---|---|
| A | L | W | I | T |
| F | Q | C | N | Y |
| K | V | H | S | E |
| P | B | M | X | J |
| U | G | R | D | O |

(4)

| | | | | |
|---|---|---|---|---|
| A | Q | H | X | O |
| F | V | M | D | T |
| K | B | R | I | Y |
| P | G | W | N | E |
| U | L | C | S | J |

(5)

| | | | | |
|---|---|---|---|---|
| A | V | R | N | J |
| F | B | W | S | O |
| K | G | C | X | T |
| P | L | H | D | Y |
| U | Q | M | I | E |

(6)

| | | | | |
|---|---|---|---|---|
| A | F | K | P | U |
| B | G | L | Q | V |
| C | H | M | R | W |
| D | I | N | S | X |
| E | J | O | T | Y |

Formation of (2) to (5) from (1), &c.

The initial letters of each row the same, but the row for any initial letter in (2) to (5) is obtained in each case by taking the positive diagonal corresponding to that initial letter in the previous arrangement, (6) from (1) by substituting rows for columns.

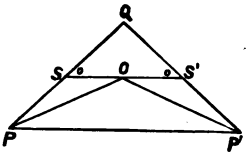
To obtain the other arrangements = Q−1, take (1) and obtain $(1_2)$, $(1_3)$, $(1_4)$, $(1_5)$ from it successively, thus $(1_2)$ from (1). The first row remains the same; in the second, third, fourth, fifth, remove each letter *one, two, three, four* places to the right. Similarly $(1_3)$ from $(1_2)$, &c.; lastly (1) from $(1_5)$ as is obvious; therefore Q = 5.

---

**11203.** (Professor B. O. PEIRCE.)—PP′ is an arc of continuous curvature. The tangents drawn to the arc at its extremities cut the tangent which is parallel to the chord PP′, at S and S′ respectively. Show that, as PP′ is made to approach zero, the ratio of SS′ to PP′ approaches the limit $\frac{1}{2}$.

---

*Solution by* Profs. CURTIS, M.A., MALILAL MALIK, *and others.*




In such a curve those qualities which differ to a finite degree at points which are at a finite distance, will differ to an infinitely small degree at points which are at an infinitely small distance. Hence, if O, P, P′ be three points on the curve, PS, P′S′, SS′, tangents at these points, and OP = OP′, the angles between the tangents S, and S′, are equal. Similarly, ∠SPO = ∠SOP = ∠S′P′O = ∠S′OP′ and ∠P = P′. In the limit, rejecting small quantities of the second degree, PS + SO = PO and P′S′ + S′O = P′O; therefore

$$PP' = PO + OP' = PS + SO + OS' + S'P' = 2SS'.$$

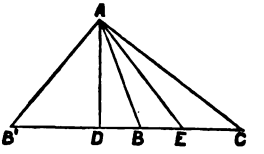
Digitized by Google

Moreover, SS′ is parallel to PP′. For ∠ SOP = ∠ SPO, and ∠ S = ∠ SPP′; therefore ∠ SOP = ∠ OPP. If OP and OP′ were at a finite distance, the angles S and S′ would only differ by an infinitely small quantity; therefore, when next each other, &c.

---

**5345.** (Dr. Hart.)—Find a triangle whose sides, perpendicular, line bisecting the vertical angle, and area, shall be integers.

*Solution by* D. Biddle.




This necessitates the finding of three right-angled triangles with integral sides and one side in common. Two of them are readily supplied by the well-known triangle whose sides are as 3 : 4 : 5. Our object, therefore, is to discover some multiple of 12 (= 3 . 4), which shall coincide with one of the legs of another right-angled triangle having integral sides of a different ratio, so as to complete the series of integral lines required, and fulfil the conditions.

Trying, first, the case in which AB′ and AC are on opposite sides of AD, let AD = 3 . 4 . $x$, AB′ = 3 . 5 . $x$, AC = 4 . 5 . $x$, B′D = 3 . 3 . $x$, DC = 4 . 4 . $x$; it is required to find a right-angled triangle, having AD as one of its legs, and an integral hypotenuse, say AB, such that B′B : BC = AB′ : AC, in which case AB will bisect ∠ B′AC. Let

$$AB - AD = n; \text{ then } DB = (2n \,.\, 3 \,.\, 4 \,.\, x + n^2)^{\frac{1}{2}},$$

and $$3 \,.\, 3 \,.\, x + (2n \,.\, 3 \,.\, 4 \,.\, x + n^2)^{\frac{1}{2}} : 4 \,.\, 4 \,.\, x - (2n \,.\, 3 \,.\, 4 \,.\, x + n^2)^{\frac{1}{2}} = 3 \,.\, 5 \,.\, x : 4 \,.\, 5 \,.\, x;$$

whence we obtain $x = n\,(49 \pm 35\sqrt{2})/12$.

Consequently, the conditions cannot be fulfilled in this way.

Trying, next, the case in which AB and AC (as sides of the triangle) are on the same side of AD, and AE (= AB) bisects the angle at the apex, let AD = 3 . 4 . $x$, AC = 4 . 5 . $x$, AE = 3 . 5 . $x$, and, also, as before, DB = $(2n \,.\, 3 \,.\, 4 \,.\, x + n^2)^{\frac{1}{2}}$. Then,

$$3 \,.\, 3 \,.\, x - (2n \,.\, 3 \,.\, 4 \,.\, x + n^2)^{\frac{1}{2}} : 4 \,.\, 4 \,.\, x - 3 \,.\, 3 \,.\, x = 3 \,.\, 4 \,.\, x + n : 4 \,.\, 5 \,.\, x;$$

whence $x = n\,(19 \pm 20)/32$. Consequently, if we make $n = 32$, we can take $x = 39$. Then AD = 468, AB = 500, AC = 780, BC = 448, AE = 585, and the area of ABC = 104832. As subsidiary values, we have DB = 176, BE = 175, EC = 273.

---

**11253.** (Professor Neuberg.)—Soient $u_1$, $u_2$, $u_3$, $u_4$ quatre droites situées dans un même plan. Dans les triangles $u_1u_2u_3$, $u_1u_3u_4$, $u_1u_4u_2$ on mène les médianes qui aboutissent aux milieux des côtés situés sur $u_1$.

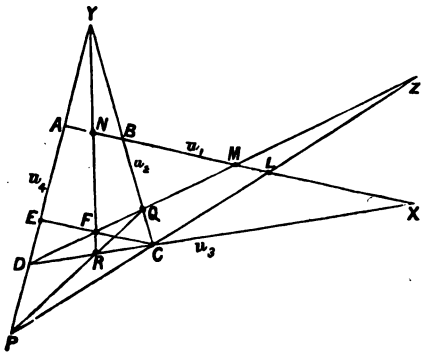
Digitized by Google

Démontrer que les points où ces médianes rencontrent respectivement $u_4$, $u_2$, $u_3$ sont en ligne droite.

---

*Solution by* J. Rice, M.A.; R. Knowles, B.A.; *and others.*

The lines $u_1$, $u_2$, $u_3$, $u_4$ are represented as in the figure by ABX, CBY, DCX, DAY. L, M are mid-points of BX, AX.




CL, DM meet $u_4$, $u_2$ in P, Q; and PQ meets $u_3$ in R.

Join YR, cutting $u_1$ in N and DM in F. DM and CL meet in Z.

Then the pencil (C. ZQFD) is harmonic; therefore, since BX is bisected at L, BX is parallel to CE; therefore, since DM is the median of $\triangle$DAX, CF = FE; therefore AN = NB; therefore, &c.

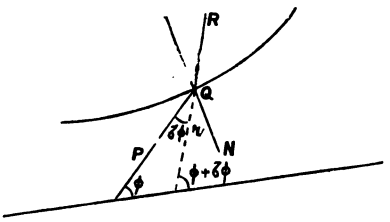
---

**5321.** (Professor Hudson, M.A.)—If the path of a ray cut at a constant angle $\alpha$ the surfaces of equal density in a variable medium, prove that $\mu = \mu_0 e^{\phi \tan \alpha}$, where $\phi$ is the inclination of the path to a fixed line.

---

*Solution by the* Proposer.

Let QN be normal to the surface of equal refractive index $\mu$; PQR the path of the ray in the neighbourhood of Q; $\phi$ the inclination of QP to a fixed line; $\phi + \delta\phi$ the inclination of RQ to that fixed line. We take the refractive index on the P side of Q to be $\mu$ and on the R side of Q to be $\mu + \delta u$, and ultimately make $\delta\mu$, $\delta\phi$






Digitized by Google

vanish. $PQN = 90 - \alpha$; $\therefore \cos\alpha = \left(1 + \frac{\delta\mu}{\mu}\right)\cos(\alpha + \delta\phi)$;

therefore $$0 = \frac{\delta\mu}{\mu}\cos\alpha - \delta\phi \,.\, \sin\alpha + \&c.,$$

whence $\frac{1}{\mu}\frac{d\mu}{d\phi} = \tan\alpha$; therefore $\frac{\mu}{\mu_0} = \epsilon^{\phi\tan\alpha}$, if $\mu = \mu_0$ when $\phi = 0$.

---

**11221.** (Professor CATALAN.)—Intégrer l'équation

$$y' = (xy - y^2)/(1 - x^2).$$

---

*Solution by* H. W. CURJEL, B.A.; Prof. CHAKRAVARTI; *and others.*

$$-\frac{dz}{dx} = \frac{xz-1}{1-x^2}, \quad \text{where } z = \frac{1}{y}; \quad \therefore \frac{dz}{dx} + z\frac{x}{1-x^2} = \frac{1}{1-x^2};$$

$$\therefore z = Ce^{-\int\frac{x}{1-x^2}dx} + e^{-\int\frac{x}{1-x^2}dx}\int\frac{1}{1-x^2}e^{\int\frac{x}{1-x^2}dx}dx$$

$$= C(1-x^2)^{\frac{1}{2}} + (1-x^2)^{\frac{1}{2}}\int\frac{1}{(1-x^2)^{\frac{3}{2}}}dx$$

$$= (1-x^2)^{\frac{1}{2}}\left\{C + \frac{x}{(1-x^2)^{\frac{1}{2}}}\right\}.$$

---

**11251.** (Professor DE LONGCHAMPS.)—Soit P une parabole donnée de foyer F. D'un point M, arbitrairement choisi dans son plan, on mène à cette courbe les normales MA, MB, MC. Les tangentes aux points A, B, C forment un triangle A'B'C'. Démontrer que la circonférence A'B'C' est vue du point M sous un angle constant $\alpha$ tel que $\sin\frac{1}{2}\alpha = \frac{1}{3}$.

---

*Solution by* Prof. RAMASWAMI AIYAR; Madame THÉRÈSE; *and others.*

If tangents be drawn from $(x, y)$ to $y^2 = 4ax$, the normals at the points of contact intersect at $(x', y')$ is $ax' = y^2 - ax + 2a^2$ and $ay' = -xy$. Now, supposing $(x', y')$ to be the point M, these equations determine the points A', B', C'.

Multiply the first by $-x/a$, and reduce by the second; thus

$$-xx' = yy' + x^2 - 2ax;$$

adding this to the first, we see that A', B', C' lie upon the circle

$$y(y+y') + x(x+x'-2a) = 0.$$

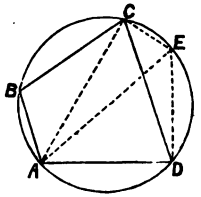
(This result is given in Dr. CASEY's *Conics*, p. 161, Quest. 58.) It is now easy to show that the square on the tangent from $x'y'$ = eight times the square on the radius; which proves the neat property pointed out.

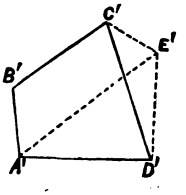
Digitized by Google

**11194.** (R. Chartres.)—Give a simple proof, without infinitesimal changes, that when four straight lines are given, the area enclosed will be a maximum when the figure is concyclic.

*Solution by* Prof. Zerr; *the* Proposer; *and others.*

Let ABCD be the quadrilateral inscribed in a circle, A′B′C′D′ a quadrilateral equilateral with ABCD, but not inscribable in a circle. Draw



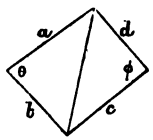




the diameter AE, connect CE, ED; on C′D′ construct the triangle C′E′D′ equal to CED. Draw AC. Now in the polygon ABCE, AB, BC, CE are given, and ABC will remain the same whether ∠ ACE be increased or diminished. But ACE is a maximum when ∠ ACE is a right angle.

Therefore of all polygons formed of sides all given but one, the polygon inscribed in a semicircle, having the undetermined side for its diameter, is the maximum; therefore ABCE > A′B′C′E′, ADE > A′D′E′; therefore ABCED > A′B′C′E′D′; but CED = C′E′D′; therefore ABCD > A′B′C′D′.

[The Proposer's proof is as follows:—

Let $\frac{1}{2}ab = p$ and $\frac{1}{2}cd = q$;




then (1) area $= p \sin\theta + q \sin\phi$,

and (2) $p\cos\theta - q\cos\phi =$ constant,

from values of diagonal. Now, when (1) is a maximum, so will $(1)^2 + (2)^2$ or $p^2 + q^2 - 2pq\cos(\theta + \phi)$; therefore $\theta + \phi = \pi$.]

**10930.** (Professor Curtis, S.J., M.A.)—Prove that six conics may be drawn osculating each other at a given point, and each osculating a given conic.

*Solution by the* Proposer.

This may be proved by projecting Casey's theorem, that

"Six circles may be drawn through any point, osculating a given conic in whose plane the point is, &c."

**11247.** (H. Briggs, M.A.)—Given AC = 1000 links, ∠ ACD = 20°, ∠ DCB = 35°, find the angle CAB, so as to cut off an area CDB = 2 acres.

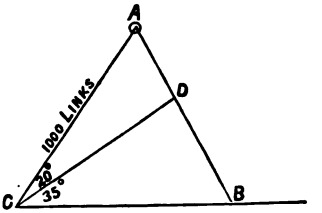
Digitized by Google

*Solution by* Rev. T. R. Terry, M.A.; M. Brierley; *and others.*

Let $\theta$ be the angle required; then

$$5\sin^2\theta\sin 35^\circ = 2\sin(\theta+20^\circ)\sin(\theta+55^\circ),$$

or $$f(\theta) = \mathrm{L}\sin(\theta+20^\circ) + \mathrm{L}\sin(\theta+55^\circ) - 2\mathrm{L}\sin\theta = \cdot 156531.$$




From a figure we see that $\theta$ lies between 0° and 125°, and that $f(\theta)$ decreases as $\theta$ increases; and on trial we see that the root lies between 50° and 60°, and the theory of proportional parts leads us to try the following:—

when $\theta = 52^\circ\,47'\,15''$, then $f(\theta) = \cdot 156567$:

when $\theta = 52^\circ\,47'\,30''$, then $f(\theta) = \cdot 156518$;

therefore the required root is $52^\circ\,47'\,26''$.

---

**11167.** (Professor Mannheim.)—Un quadrilatère articulé ABCD a son sommet A qui est fixe; les côtés AB, AD tournent autour de A d'angles égaux en sens inverses. On demande le lieu du point C.

---

*Solution by* H. J. Woodall; Prof. Gophar; *and others.*

$AB = a$, $BC = b$, $CD = c$, $DA = d$,

$$BAy = \theta = yAD;$$

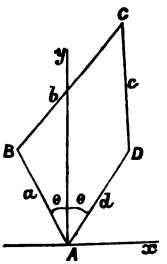
$B(-a\sin\theta, a\cos\theta)$, $D(d\sin\theta, d\cos\theta)$;

let C be $(x, y)$; therefore

$$BC = b = [(x + a\sin\theta)^2 + (y - a\cos\theta)^2]^{\frac{1}{2}};$$

therefore $$x^2 + y^2 + 2ax\sin\theta - 2ay\cos\theta + a^2 - b^2 = 0;$$

$$x^2 + y^2 - 2dx\sin\theta - 2dy\cos\theta + d^2 - c^2 = 0.$$




Eliminate $\sin\theta$ and $\cos\theta$; we have the sextic

$$(4ad)^2 = \frac{1}{y^2}\left\{(a+d)(x^2+y^2) + d(a^2-b^2) + a(d^2-c^2)\right\}^2 + \frac{1}{x^2}\left\{(a-d)(x^2+y^2) - d(a^2-b^2) + a(d^2-c^2)\right\}^2.$$

[The Proposer remarks that—"Si le quadrilatère ABCD est un parallélogramme, la courbe décrite par C est simplement une ellipse."]

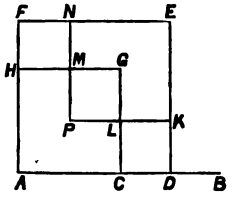
---

**11366.** If a line AB is bisected in C and divided unequally in D either internally or externally, to prove that the sum of the squares on AD, DB is equal to twice the sum of the squares on AC, CD.

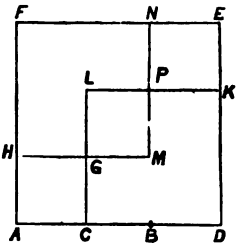
Digitized by Google

*Solution by* Miss HILDA HUDSON.

On AD, AC, CD draw squares ADEF, ACGH, CDKL, all on the same side of AB; on FH, within the square AE, describe square




II. 9.




II. 10.

FHMN; this is equal to the square on CD. Let NM, KL, produced if necessary, meet in P.

Then PE is equal to the square on AC, and PG to the square on BD. The sum of the squares on AD, DB is equal to the figures AE, PG; that is, to AG, PE, CK and FM; that is, to twice the sum of AG and CK; that is, to twice the sum of the squares on AC, CD.

[The foregoing proof of Euc. II., 9, 10, due to a girl 10 years old, has been communicated to the London Mathematical Society.]

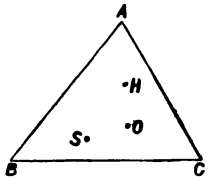
---

**10879.** (Professor GENESE, M.A.)—With any point O in the plane of an ellipse as centre, prove that (1) two real circles can be drawn, in either of which triangles may be inscribed whose sides touch the ellipse; and (2) the radius R of either is given by

$$(R^2-OS^2)(R^2-OH^2) = 4b^2R^2.$$

---

*Solution by* Profs. RAMASWAMI AIYAR, ZERR, *and others.*




Let ABC = Δ be a triangle whose circumcentre is O, and whose sides $a$, $b$, $c$ touch the ellipse (semi-axis minor = $\gamma$), Let $t_1$, $t_2$ denote the tangents from A to the circles about

the triangles BSC = $\Delta_1$

and BHC = $\Delta_2$.

Now the determinant, where the $xy$'s denote A, B, C, and S, is equal to both the members of the following relation:—

$$\begin{vmatrix} x_1^2+y_1^2, & x_1 & y_1 & \frac{1}{2} \\ x_2^2+y_2^2, & x_2 & y_2 & \frac{1}{2} \\ x_3^2+y_3^2, & x_3 & y_3 & \frac{1}{2} \\ x_4^2+y_4^2, & x_4 & y_4 & \frac{1}{2} \end{vmatrix}.$$

Therefore $(R^2-OS^2)\,\Delta = t_1^2 \cdot \Delta_1$; similarly $(R^2-OH^2)\,\Delta = t_2^2 \cdot \Delta_2$.

Digitized by Google

Again, S, H being isogonal conjugates with respect to ABC, it is easy to show that $t_1 . t_2 = bc$; also $\Delta_1 . \Delta_2 = \frac{1}{4}a^2\gamma^2$.

Therefore $(R^2 - OS^2)(R^2 - OH^2) = 4\gamma^2 \left(\frac{abc}{4\Delta}\right)^2 = 4\gamma^2 R^2$;

and the values of $R^2$ are both positive.

[Mr. HEPPEL proves the theorem thus:—To avoid confusion of letters, replace $b$ by $k$. In such a system, let $\alpha$, $\beta$, $\gamma$ be the coordinates of S, referred to the triangle, and $\alpha'$, $\beta'$, $\gamma'$ those of H. Those of O are R cos A, R cos B, R cos C. Then $OS^2 = -(R/\Delta)\,\Sigma a\,(R\cos B - \beta)(R\cos C - \gamma)$;

$\therefore\ \Delta . OS^2 = -R^2 . \Sigma a \cos B \cos C + R^2 . \Sigma a\,(\beta \cos C + \gamma \cos B) - R . \Sigma a\beta\gamma$.

But $$a\alpha' = \beta\beta' = \gamma\gamma' = k^2,$$

and $$\Sigma a \cos B \cos C = \Delta/R, \quad \Sigma a\,(\beta \cos C + \gamma \cos B) = 2\Delta,$$

$$\Sigma a\beta\gamma = a\beta\gamma/k^2 . \Sigma a\alpha' = 2\Delta/k^2 . a\beta\gamma,$$

therefore $$R^2 - OS^2 = 2a\beta\gamma/k^2 . R.$$

Similarly $R^2 - OH^2 = 2\alpha'\beta'\gamma'/k^2 . R$, $\therefore\ (R^2 - OS^2)(R^2 - OH^2) = 4k^2R^2$,

and this equation must give two real positive values of $R^2$ for any value of $k$. [For another solution, see Vol. LV., p. 97.]

---

**11186.** (H. MACCOLL, B.A.)—From the conditional statement, "If all Z is Y, all X is Y," may we logically infer that "all X is Z"?

---

*Solution by* Professor CURTIS, S.J., M.A.

The equivalent of the conditional statement is: Suppose that all Z is Y, therefore all X is Y. If these form a logical sequence, then "all X is Z" must be the suppressed premiss.

[The PROPOSER remarks that he does not quite agree with Professor CURTIS as to the "suppressed premiss." To take an illustration, suppose we are speaking of stereotyped plates and the impressions obtained from them. May we not infer that, if all the *plates* are *incorrect*, all the *impressions* from them must also be *incorrect*? And is it necessarily a suppressed premiss that *all the impressions are plates*? Of course premisses of that kind *are* often suppressed, and purposely, when truth is not the object aimed at. But this is a digression. It would appear that in this case the statement, "All X is Z," is neither admissible as a premiss nor legitimate as a conclusion.]

---

**10806.** (The late Professor WOLSTENHOLME, Sc.D.)—Prove that, if $x$, $y$, $z$ be three finite quantities whose sum is zero, and if

$(ax + by + cz)^2 = 4\,(bc + ca + ab)(yz + zx + xy)$ and $ax^2 + by^2 + cz^2 = 0$,

then will $$(a + b + c)(bc + ca + ab) = 9abc.$$

Digitized by Google

*Solution by* Rev. T. R. Terry, M.A.

From (1) and (3), $xy(a+b)+yz(b+c)+zx(c+a)=0$.
From (1) and (2), $xy(a+b)(a+b+4c)+\ldots=0$;
hence, solving, we get $\frac{(a+b)xy}{a-b}=\frac{(b+c)yz}{b-c}=\frac{(c+a)zx}{c-a}$
whence, by (1), $\frac{a+b}{a-b}+\frac{b+c}{b-c}+\frac{c+a}{c-a}=0$;
which, cleared of fractions, gives the required result.
[For another proof, see Vol. LIV., p. 126.]

---

**11159.** (Professor Neuberg.)—Sommer la série

$$\frac{x}{1}-\frac{x^2}{2}+\frac{x^5}{5}-\frac{x^6}{6}+\frac{x^9}{9}-\frac{x^{10}}{10}+\frac{x^{13}}{13}-\ldots.$$

---

*Solution by* H. E. Soper, M.A.; R. Knowles, B.A.; *and others.*

Writing the series $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots=\log(1+x)$,

$$x-\frac{x^3}{3}+\ldots=\tan^{-1}x,\quad \frac{x^2}{2}-\frac{x^4}{4}\ldots=\log(1+x^2);$$

it is easily seen that the given series is obtained by adding the first two of the above, subtracting the third, and dividing by 2.

---

**11156.** (The late Professor Wolstenholme, Sc.D.)—Prove that, if $r$ be any positive integer, $n>1$, and $\theta\equiv\pi/n$,

$$\int_0^\infty\frac{(\log x)^r}{1+x^n}dx=\theta^{r+1}\frac{d^r}{d\theta^r}(\operatorname{cosec}\theta),\quad \int_0^\infty\frac{(\log x)^r}{x^n-1}dx=\theta^{r+1}\frac{d^r}{d\theta^r}(\cot\theta).$$

[These are given in the Proposer's book of problems, page 333, (41), (42), with the unnecessary restriction of $r$ being odd.]

---

*Solution by* H. E. Soper, M.A.; Professor Madhavarao; *and others.*

From the integral $\int_0^\infty\frac{x^{a-1}}{1+x}dx=\pi\operatorname{cosec}a\pi$,

it may be shown that $\int_0^\infty\frac{x^y}{1+x^n}dx=\frac{\pi}{n}\operatorname{cosec}\frac{1+y}{n}\pi$,

whence, expanding in powers of $y$ and equating coefficients of $y^r$,

$$\int_0^\infty\frac{(\log x)^r}{1+x^n}dx=\theta^{r+1}\frac{d^r}{d\theta^r}(\operatorname{cosec}\theta),\text{ where }\theta\equiv\pi/n.$$

Digitized by Google

[Professor Zerr solves the Question, completely, thus:—

$$\int_0^\infty \frac{dx}{1+x^n} = \frac{\pi}{n \sin \pi/n} = \theta \operatorname{cosec} \theta ;$$

differentiating successively with respect to $n$, we get

$$\int_0^\infty \frac{\log x}{1+x^n}\,dx = \theta^2 \frac{d}{d\theta}(\operatorname{cosec}\theta), \quad \int_0^\infty \frac{(\log x)^2}{1+x^n}\,dx = \theta^3 \frac{d^2}{d\theta^2}(\operatorname{cosec}\theta),$$

$$\ldots\ldots, \quad \int_0^\infty \frac{(\log x)^r}{1+x^n}\,dx = \theta^{r+1} \frac{d^r}{d\theta^r}(\operatorname{cosec}\theta).$$

Similarly, $$\int_0^\infty \frac{dx}{x^n-1} = \frac{\pi}{n} \cot \frac{\pi}{n} = \theta \cot \theta,$$

$$\int_0^\infty \frac{(\log x)}{x^n-1}\,dx = \theta^2 \frac{d}{d\theta}(\cot\theta), \quad \int_0^\infty \frac{(\log x)^2}{x^n-1}\,dx = \theta^3 \frac{d^2}{d\theta^2}(\cot\theta),$$

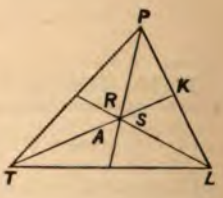
$$\ldots\ldots, \quad \int_0^\infty \frac{(\log x)^r}{x^n-1}\,dx = \theta^{r+1} \frac{d^r}{d\theta^r}(\cot\theta).]$$

---

**10863.** (Rev. T. Roach, M.A.)—Parabolas are drawn, each touching two sides of an equilateral triangle, where they meet the third side; prove that (1) each pair meet on the axis of the third parabola; and (2) the square of the common chord is to the square of the common tangent in the ratio of 16 : 27.

*Solution by the* Proposer.

TK is axis of one parabola; $\therefore$ TS = SP, therefore S is the focus. Similarly S is the focus of the two other parabolas. Let two of the parabolas cut in R, then RS bisects TSP by symmetry; therefore R lies on SL.




Also ST = $\frac{2}{3}$TK, AT = $\frac{1}{3}$TK, $\therefore$ AS = $\frac{1}{6}$TK.

And SR = 2SN, $\therefore$ 2SN = XN = 2AS − SN,

$\therefore$ SN = $\frac{2}{5}$AS, $\therefore$ RS = $\frac{4}{5}$AS = $\frac{2}{9}$TK.

$\therefore$ LR² : TP² = ($\frac{2}{9}$TK + SL)² : $\frac{4}{9}$TK²,

= ($\frac{2}{9}$TK + $\frac{2}{3}$TK)² : $\frac{4}{9}$TK² = $\frac{64}{81}$ : $\frac{4}{9}$ = 16 : 27.

---

**11238.** (H. F. Davis, M.A.)—From the multiplication of A + $x$ by B + $y$, deduce a simple rule for calculating such values as 21$\frac{1}{8}$ yards at 19$\frac{3}{4}$ pence each, or 16$\frac{5}{8}$ yards at 2*s.* 3$\frac{1}{2}$*d.* each.

*Solution by the* Proposer.

It is generally believed that when the City man learns for the first time that his "young hopeful" has been promoted to the cultivation of

Digitized by Google

the science of Algebra, *vice* Vulgar Fractions and Rule of Three superseded, he is somewhat apt to descant upon the unnecessary acquirement of knowledge unfitting for a commercial career.

Let us, however, suppose the boy, his schooling over, applies for employment in the wholesale. Having satisfactorily disposed of the usual *viva voce* test of "$3\frac{1}{4}$ at $7\frac{1}{2}d.$" he is forthwith installed in the entering desk. There he will constantly come across a certain type of "extension" which a simple application of a well-known algebraical formula will enable him very easily to calculate or check.

The formula is as follows:—If $x$, $y$ be very small compared with A, B, then in the product of $A+x$ and $B+y$, which is $AB+Ay+Bx+xy$, the last term $xy$ may be omitted altogether.

Thus $21\frac{7}{8}$ yards at $19\frac{3}{4}d. = (22-\frac{1}{8})(20-\frac{1}{4}) = 440-\frac{20}{8}-\frac{22}{4} = 440-2\frac{1}{2}-5\frac{1}{2}$ $= 432d. = £1.\ 16s.$, which is correct to $\frac{1}{32}$ of a penny. Again,

$$16\tfrac{5}{6} \text{ yards at } 2/3\tfrac{1}{2} = (16\tfrac{1}{2}+\tfrac{1}{6})(28-\tfrac{1}{2}) = \left.\begin{matrix}168\\280\\14\end{matrix}\right\} +3-8 = 457d. = £1.\ 18s.\ 1d.$$

---

**11255.** (Professor MALET, F.R.S.)—Prove the following method of finding the equation of squared differences of the roots of

$$f(x) \equiv a_0x^n + a_1x^{n-1} + \&c. = 0.$$

Let $$f(x+\tfrac{1}{2}\sqrt{z}) \equiv \phi(x, z) + \sqrt{z}\,\psi(x, z),$$

where $\phi$ and $\psi$ are rational integral functions of $x$ and $z$; then, if we eliminate $x$ between $\phi = 0$ and $\psi = 0$, the resulting equation in $z$ is the required equation. Apply this method to the cubic.

---

*Solution by* Professor GENESE, M.A.; H. G. DAWSON, M.A.; *and others.*

Since also $f(x-\frac{1}{2}\sqrt{z}) = \phi(x, z) - \sqrt{z}\,\psi(x, z)$, the elimination is the same as that of $\sigma$ from $f(\sigma+\frac{1}{2}\sqrt{z}) = 0$, $f(\sigma-\frac{1}{2}\sqrt{z}) = 0$. Now, if $\sigma = \frac{1}{2}(\alpha+\beta)$, $\sqrt{z} = \alpha-\beta$, $\alpha$ and $\beta$ being roots of the equation, the last equations become $f(\alpha) = 0$, $f(\beta) = 0$; therefore, &c.

[The PROPOSER states that this solution is incorrect, the elimination in the question being *not* the same as that in the solution. The eliminant in the latter case contains as a factor $z^{\frac{1}{2}n}$, and the point of the question, which has been ingeniously missed, was to show that this irrelevant factor does not enter the result if the elimination as proposed is performed.]

---

**1069.** (The late Professor WOLSTENHOLME, D.Sc.)—The mean value of $(xyz)^{\frac{1}{2}}$, where $x$, $y$, $z$ are areal coordinates of a point within the triangle of reference, is $\frac{4\pi}{105}$; and the mean value of $(wxyz)^{\frac{1}{2}}$, where $w$, $x$, $y$, $z$



Digitized by Google

are tetrahedral coordinates of a point within the tetrahedron of reference, is $\frac{\pi^2}{320}$. Also the mean value of $(wxyz)^{n-1}$ is $6\{\Gamma(n)\}^4 \div \Gamma(4n)$.

---

*Solution by* Professor ZERR, M.A.

In the first case we have the condition $x+y+z=1$.
In the second and third cases the condition is $x+y+z+w=1$.
Therefore in the first, second, third cases, the respective mean values are

$$M = \frac{\iiint x^{\frac{1}{2}} y^{\frac{1}{2}} z^{\frac{1}{2}}\, dx\, dy\, dz}{\iiint dx\, dy\, dz} = \frac{\{\Gamma(\frac{3}{2})\}^3}{\Gamma(\frac{9}{2})} \Big/ \frac{1}{\Gamma(3)} = \frac{4\pi}{105};$$

$$M_1 = \frac{\iiint x^{\frac{1}{2}} y^{\frac{1}{2}} z^{\frac{1}{2}} w^{\frac{1}{2}}\, dx\, dy\, dz\, dw}{\iiint dx\, dy\, dz\, dw} = \frac{\{\Gamma(\frac{3}{2})\}^4}{\Gamma(6)} \Big/ \frac{1}{\Gamma(4)} = \frac{\pi^2}{320};$$

$$M_2 = \frac{\iiint x^{n-1} y^{n-1} z^{n-1} w^{n-1}\, dx\, dy\, dz\, dw}{\iiint dx\, dy\, dz\, dw} = \frac{\{\Gamma(n)\}^4}{\Gamma(4n)} \Big/ \frac{1}{\Gamma(4)} = \frac{6\{\Gamma(n)\}^4}{\Gamma(4n)};$$

also the mean values of $(x_1, x_2, x_3, \ldots x_n)^{\frac{1}{2}}$, and of $(x_1, x_2, x_3, \ldots x_n)^{r-1}$, for all positive values of $x_1, x_2, \ldots$, such that $x_1 + x_2 + \ldots + x_n = 1$, are

$$M_3 = \frac{\iiint \ldots x_1^{\frac{1}{2}} \quad \ldots x_n^{\frac{1}{2}}\, dx_1\, dx_2 \ldots dx}{\iiint \ldots dx_1\, dx_2 \ldots dx_n} = \frac{\{\Gamma(\frac{3}{2})\}^n}{\Gamma(\frac{3}{2}n)} \Big/ \frac{1}{\Gamma(n)}$$

$$= \Gamma(n)\{\Gamma(\tfrac{3}{2})\}^n / \Gamma(\tfrac{3}{2}n);$$

$$M_4 = \frac{\iiint \ldots x_1^{r-1} x_2^{r-1} \ldots x_n^{r-1}\, dx_1\, dx_2 \ldots dx_n}{\iiint \ldots dx_1\, dx_2 \ldots dx_n}$$

$$= \frac{\{\Gamma(r)\}^n}{\Gamma(nr)} \Big/ \frac{1}{\Gamma(n)} = \frac{\Gamma(n)\{\Gamma(r)\}^n}{\Gamma(nr)}.$$

---

**11009.** (The late Professor CLIFFORD, F.R.S.)—A bubble consists of $v$ cubic centimetres of water, and requires an amount of work $w$ to increase its radius from 10 to 20 centimetres; if a cubic centimetre of water requires the work $h$ to convert it into steam at the same temperature, find the thickness of the film when the liquid may be regarded as just passing

Digitized by Google

into the gaseous form, the bubble being blown out into a state of extreme tenuity.

---

*Solution by* H. W. Curjel, B.A.

Let $r$ be the external and $r'$ the internal radius of the bubble; and $t$ be the surface tension of the water; then (1) $\frac{4}{3}\pi(r^3-r'^3) = v$;

$$w = \int_{r=10}^{r=20}\left(4\pi r^2\frac{2t}{r}dr + 4\pi r'^2\frac{2t}{r'}dr'\right) = 4\pi t\,(r^2+r'^2)_{r=10}^{r=20}$$

$$= 4\pi t\left\{20^2-10^2+\left(20^3-\frac{3v}{4\pi}\right)^{\frac{2}{3}}-\left(10^3-\frac{3v}{4\pi}\right)^{\frac{2}{3}}\right\},$$

which determines $t$.

If $r_1$, $r'$ are the values of $r$, $r'$ when liquid is just passing into vapour,

$$vh = \int_{r=(3v/4\pi)}^{r}(8\pi t - r\,dr + 8\pi t r'\,dr') = 4\pi t\left\{r^2+r'^2-\left(\frac{3v}{4\pi}\right)^{\frac{2}{3}}\right\};$$

$$\therefore\quad r^2+r'^2 = \frac{vh}{4\pi t} + \left(\frac{3v}{4\pi}\right)^{\frac{2}{3}} = a^2 \text{ (say)} \qquad \ldots\ldots (2).$$

Let T = thickness of film; then $T = r-r'$ ..... (3).

(1) becomes $T(r^2+r'+r'^2) = \frac{3v}{4\pi} = b^3$ (say) ..... (4).

From (2), (3), (4), $a^2-T^2 = 2rr' = 2\,\frac{b^3}{T} - 2a^2$;

$$\therefore\quad T^3-3a^2T+2b^3 = 0.$$

This gives T, the required value being the root whose approximate value is $2b^3/3a^2$.

---

**10955.** (Professor Zerr.)—Find the centroid of the area between the cardioid $\gamma = \frac{1}{2}a(1+\cos\theta)$, and its first pedal, when the density at a point varies as the $n^{\text{th}}$ power of the distance of the point from the origin.

---

*Solution by the* Proposer.

From Statics, we have $\bar{x} = \dfrac{\int_\alpha^\beta\int_0^r k\,\rho r^2\cos\theta\,d\theta\,dr}{\int_\alpha^\beta\int_0^r k\,\rho r\,d\theta\,dr}$.

In this example, $k$ is constant, and $\rho = \mu r^n$; hence we have

$$\bar{x} = \frac{\int_\alpha^\beta\int_\gamma^{\gamma_1} r^{n+2}\cos\theta\,d\theta\,dr}{\int_\alpha^\beta\int_\gamma^{\gamma_1} r^{n+1}\,d\theta\,dr} = \frac{n+2}{n+3}\,\frac{\int_0^{\frac{1}{2}\pi}\gamma_1^{n+3}\cos\theta\,d\theta - \int_0^{\pi}\gamma^{n+3}\cos\theta\,d\theta}{\int_0^{\frac{1}{2}\pi}\gamma_1^{n+2}\,d\theta - \int_0^{\pi}\gamma^{n+2}\,d\theta}.$$

Digitized by Google

But $\gamma_1 = a\cos^3\frac{1}{3}\theta$, and $\gamma = a\cos^2\frac{1}{2}\theta$; thus, putting $\frac{1}{3}\theta = \phi$, $\frac{1}{2}\theta = \phi_1$,

$$\bar{x} = \frac{n+2}{n+3}\,a\,\frac{3\int_0^{\frac{1}{2}\pi}\cos^{3n+9}\phi\cos 3\phi\,d\phi - 2\int_0^{\frac{1}{2}\pi}\cos^{2n+6}\phi_1\cos 2\phi_1\,d\phi_1}{3\int_0^{\frac{1}{2}\pi}\cos^{3n+6}\phi\,d\phi - 2\int_0^{\frac{1}{2}\pi}\cos^{2n+4}\phi_1\,d\phi_1}$$

$$= \frac{n+2}{n+3}\,a\,\frac{\begin{array}{r}3\left\{4\dfrac{\Gamma[\frac{1}{2}(3n+13)]}{\Gamma[\frac{1}{2}(3n+14)]} - 3\dfrac{\Gamma[\frac{1}{2}(3n+1)]}{\Gamma[\frac{1}{2}(3n+12)]}\right\}\\ -2\left\{2\dfrac{\Gamma[\frac{1}{2}(2n+9)]}{\Gamma[\frac{1}{2}(2n+10)]} - \dfrac{\Gamma[\frac{1}{2}(2n+7)]}{\Gamma[\frac{1}{2}(2n+8)]}\right\}\end{array}}{3\dfrac{\Gamma[\frac{1}{2}(3n+7)]}{\Gamma[\frac{1}{2}(3n+8)]} - 2\dfrac{\Gamma[\frac{1}{2}(2n+5)]}{\Gamma[\frac{1}{2}(2n+6)]}}.$$

Reducing, we have

$$\bar{x} = \frac{n+2}{n+4}\,a\,\frac{\begin{array}{r}\dfrac{3(3n+7)}{(3n+10)}\Gamma[\frac{1}{2}(3n+7)]\,\Gamma(n+3)\\ -\dfrac{(2n+5)}{(n+3)}\Gamma[\frac{1}{2}(3n+8)]\,\Gamma[\frac{1}{2}(2n+5)]\end{array}}{3\Gamma[\frac{1}{2}(3n+7)]\,\Gamma(n+3) - 2\Gamma[\frac{1}{2}(3n+8)]\,\Gamma[\frac{1}{2}(2n+5)]};$$

$$\bar{y} = 0.$$

---

**9838.** (Professor Hudson.)—A given segment of a parabola, whose base is a chord perpendicular to its axis, has another such segment inscribed in it, their bases being parallel; find when the inscribed segment is maximum. If, in this last, another maximum segment be inscribed, and so on, show that, when these maximum segments become indefinitely small, they will lie at the centre of gravity of the original segment.

---

*Solution by* Professors Zerr, Bhattacharya, *and others.*

Let $y^2 = px$ be the equation to the given parabola, $a$ = its altitude, $2b$ = its base, $x$ = the distance of the base of the maximum inscribed segment from the vertex of the given parabola, and $y$ = half its base; then $p = a^{-1}b^2$; area of inscribed segment $= \frac{4}{3}y(a-x) = \frac{4}{3}(a-x)(px)^{\frac{1}{2}}$. Differentiating and solving for $x$, we get $x = \frac{1}{3}a$, $y = \frac{1}{3}b\sqrt{3}$; hence the maximum inscribed segment has for its altitude $\frac{2}{3}a$, for its base $\frac{2}{\sqrt{3}}b$.

Thus, the altitudes of the 2nd, 3rd, $n$th inscribed segments are

$$\frac{2^2}{3^2}a,\quad \frac{2^3}{3^3}a,\quad \frac{2^n}{3^n}a.$$

The distance of the centre of gravity of the given parabola from the vertex $= \frac{2}{5}a$. For the first and second inscribed segments,

$$a - \tfrac{2}{5}\times\tfrac{2}{3}a = \tfrac{3}{5}a,\quad \tfrac{1}{3}a + \tfrac{2}{5}\times\tfrac{2}{3}a = \tfrac{3}{5}a.$$

Thus, all the inscribed segments have their centres of gravity coincident with that of the given segment: hence, when the segments become indefinitely small, they lie at the centre of gravity of the given segment.

Digitized by Google

**11078.** (A. GORDON, B.A.) — Prove that, if $r < n-1$ be a positive integer

$$\frac{\cos r\alpha_1}{\sin(\alpha_1-\alpha_2)\sin(\alpha_1-\alpha_3)\ldots\sin(\alpha_1-\alpha_n)} = 0$$

$$= \Sigma \frac{\sin r\alpha_1}{\sin(\alpha_1-\alpha_2)\sin(\alpha_1-\alpha_3)\ldots\sin(\alpha_1-\alpha_n)},$$

where the $\Sigma$ refers to a cyclic interchange of the suffixes $1, 2 \ldots n$, so that the next term is $\cos r\alpha_2 / \sin(\alpha_2-\alpha_3)\sin(\alpha_2-\alpha_4)\ldots\sin(\alpha_2-\alpha_1)$.

---

*Solution by* Professor MALLIK, M.A.; *the* PROPOSER; *and others.*

$$\frac{a^r}{(a-b)(a-c)\ldots(a-b)} + \frac{b^r}{(b-c)(b-d)\ldots(b-l)(l-a)} + \&c. \equiv 0,$$

where $a, b, c$, &c. ... are $n$ numbers and $r < n$.

Substituting $a = \cos 2\alpha_1 + i \sin 2\alpha$, with corresponding values for $b, c$, &c., and equating the real and the imaginary parts to zero, we get

$$\Sigma \frac{\cos r\alpha_1}{\sin(\alpha_1-\alpha_2)\sin(\alpha_1-\alpha_3)\ldots\sin(\alpha_1-\alpha_n)} = 0,$$

and

$$\Sigma \frac{\sin r\alpha_1}{\sin(\alpha_1-\alpha_2)\sin(\alpha_1-\alpha_3)\ldots\sin(\alpha_1-\alpha_n)}.$$

[*Otherwise*:—If there is an equation of the $n^{\text{th}}$ degree $F(x) = 0$, it is known that, if $x_1, x_2, \ldots x_n$ are its roots, then

$$\Sigma\left(\frac{x_1^r}{F'(x_1)}\right) = 0,$$

provided $r$ is a positive integer less than $n-1$.

Consider an equation with roots $\cot\alpha_1, \cot\alpha_2, \ldots \cot\alpha_n$; therefore

$$\Sigma \frac{\cot^r\alpha_1}{(\cot\alpha_1-\cot\alpha_2)\ldots(\cot\alpha_1-\cot\alpha_n)} = 0 \quad (r<n-1).$$

Put $r = (n-2), (n-3), \ldots 1, 0$ in succession

$$\text{A}-\begin{cases} \therefore \Sigma \dfrac{\cos^{n-2}\alpha_1}{\sin(\alpha_1-\alpha_2)\ldots\sin(\alpha_1-\alpha_n)} = 0, \\ \Sigma \dfrac{\cos^{n-4}}{\sin(\alpha_1-\alpha_2)\ldots\sin(\alpha_1-\alpha_n)} = 0, \\ \ldots\ldots \quad \&c. \quad \ldots\ldots \end{cases}$$

and

$$\begin{cases} \Sigma \dfrac{\cos^{n-3}\alpha_1 \sin\alpha_1}{\sin(\alpha_1-\alpha_2)\ldots\sin(\alpha_1-\alpha_n)} = 0, \\ \Sigma \dfrac{\cos^{n-5}\alpha_1 \sin\alpha_1}{\sin(\alpha_1-\alpha_2)\ldots\sin(\alpha_1-\alpha_n)} = 0, \\ \ldots\ldots \quad \&c. \quad \ldots\ldots \end{cases}$$

But $\cos r\alpha_1 = A_0\cos^r\alpha_1 + A_2\cos^{r-2}\alpha_1 + A_4\cos^{r-4}\alpha_1 + \ldots,$

Digitized by Google

where $A_0, A_2, \ldots$ &c. are numerical, and

$$\sin ra_1 = \sin a_1 \{B_1 \cos^{r-1} a_1 + B_3 \cos^{r-3} a_1 + \ldots\},$$

and

$$\Sigma \frac{\cos ra_1}{\sin (a_1 - a_2) \ldots \sin (a_1 - a_n)}$$

$$= A_0 \Sigma \frac{\cos^r a_1}{\sin (a_1 - a_2) \ldots \sin (a_1 - a_n)} + A_2 \Sigma \frac{\cos^{r-2} a_1}{\sin (\ldots) \ldots \sin (\ldots)} + A_4 \Sigma + \&c.,$$

and each of these terms by the above equations is zero. Similarly,

$$\Sigma \frac{\cos ra_1}{\sin (a_1 - a_2) \ldots \sin (a_1 - a_n)} = 0.]$$

---

**10253.** (H. J. Woodall. Suggested by Quest. 10179.)—Solve (1) the cubic $x^3 + ax = b$, (2) the quintic $x^5 + ax^2 = b$, by means of a continued fraction of the form $x = K/(L + \sqrt[2]{}\ \&c.)$; and find (3) whether the form is suitable for calculation.

*Solution by the* Proposer.

(1) $x^3 + ax = b$; $\therefore\ x(x^2 + a) = b$; $x = \dfrac{b}{a + \sqrt[2]{\dfrac{b}{a + \ldots}}}$.

(2) $x^5 + ax^2 = b$; $\therefore\ x^2(x^3 + a) = b$; $\therefore\ x = \sqrt{\dfrac{b}{a + \sqrt[3]{\dfrac{b}{a + \ldots}}}}$.

(3) Neither form is suitable for calculation.

---

**10503.** (Alpha.)—Find the equation of an ellipse passing through the five points $(x_1y_1)$, $(x_2y_2)$, $(x_3y_3)$, $(x_4y_4)$, $(x_5y_5)$.

*Solution by* H. J. Woodall.

By ordinary elimination we obtain such an equation in determinant form thus $[x^2, x_1y_1, y_2^2, x_3, y_4, 1] = 0$.

The expanded form may be obtained by the consideration that

$$12 \times 34 = k \,.\, 14 \times 23$$

includes all conics through the points 1, 2, 3, 4. $k$ can be determined so that 5 lies on the conic also.

---

**8403.** (Prof. Mukhopâdhyây, M.A., F.R.A.S.)—"Observations on the motions of the sun-spots have also established the fact that the sun

Digitized by Google

is not strictly a fixed body around which the earth revolves, but that it has a motion of its own through space." (HUXLEY's *Physiography*, p. 365.) Explain how the above fact is determined by observations of sun-spots.

*Solution by* H. J. WOODALL.

The statement is without foundation if it refers to "proper motion." The motions of the sun-spots can only furnish facts for the motion of the sun.

---

**11072.** (Rev. T. R. TERRY, M.A.)—Simplify the expression

$$(bc+ad)^3(b^3-c^3)(a^3-d^3)+(ca+bd)^3(c^3-a^3)(b^3-d^3)$$
$$+(ab+cd)^3(a^3-b^3)(c^3-d^3).$$

*Solution by the* PROPOSER.

The expression ($\equiv$ E) remains unchanged when $a$, $b$, $c$ are changed cyclically, and it vanishes when $b=c$, and when $a=d$; therefore

$$(b-c)(c-a)(a-b)(a-d)(b-d)(c-d)\equiv\Delta$$

is a factor; therefore $E=\lambda\Delta$, where $\lambda$ is of six dimensions in $a$, $b$, $c$, $d$.

Also, if $a$, $b$, $c$, $d$ be changed cyclically, both E and $\Delta$ change sign; therefore $\lambda$ remains unaltered.

Now, E vanishes when $a=0$; therefore $\lambda=abcd\{A(\Sigma a)^3+B\Sigma(ab)\}$, where A and B are constants.

Putting $a=1$, $b=-1$, $c=2$, $d=2$, we get $B=3$;

putting $a=1$, $b=-2$, $c=2$, $d=4$, we get $A=0$;

$\therefore$ $E=3abcd(b-c)(c-a)(a-b)(a-d)(b-d)(c-d)$
$$\times(ab+ac+ad+bc+bd+cd).$$

---

**9115.** (Professor IGNACIO BEYENS.)—Si D est le diamètre de la sphère circonscrite à un tétraèdre equifacial, $\alpha$, $\beta$, $\gamma$ les angles sous lesquels on voit du centre de la sphère des arêtes du tétraèdre, et V le volume du tétraèdre; on aura $V=\frac{1}{3}D^3\cos\frac{1}{2}\alpha\cos\frac{1}{2}\beta\cos\frac{1}{2}\gamma$.

*Solution by the* PROPOSER.

Nous avons $a$, $b$, $c$, étant les arêtes

$$(1)\quad V=\tfrac{1}{12}\{2(b^2+c^2-a^2)(a^2+c^2-b^2)(a^2+b^2-c^2)\}^{\frac{1}{2}},$$

mais R étant le rayon de la sphère $8R^2=a^2+b^2+c^2$, et d'ici on déduira

$$b^2+c^2-a^2=8R^2-2a^2=8R^2-8R^2\sin^2\tfrac{1}{2}\alpha=8R^2\cdot\cos^2\tfrac{1}{2}\alpha,$$

et de la même manière $a^2+c^2-b^2=8R^2\cdot\cos^2\frac{1}{2}\beta$, $a^2+b^2-c^2=8R^2\cdot\cos^2\frac{1}{2}\gamma$ (parceque $a=2R\sin\frac{1}{2}\alpha$, $b=2R\sin\frac{1}{2}\beta$, $c=2R\sin\frac{1}{2}\gamma$), et en vertu de (1)

$$V=\tfrac{1}{12}\{2\cdot2^9\cdot R^6\cdot\cos^2\tfrac{1}{2}\alpha\cos^2\tfrac{1}{2}\beta\cos^2\tfrac{1}{2}\gamma\}^{\frac{1}{2}}=\tfrac{2^5}{12}R^3\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\cos\tfrac{1}{2}\gamma$$
$$=\tfrac{1}{3}(8R^3)\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\cos\tfrac{1}{2}\gamma=\tfrac{1}{3}D^3\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\cos\tfrac{1}{2}\gamma.$$

Digitized by Google

**11002.** (Professor DE LONGCHAMPS.)—On donne deux points fixes O, A (OA = R). Soit Δ une droite mobile; on abaisse, des points O et A, sur Δ, des perpendiculaires OO′, AA′. Ayant posé OO′ = $u$, AA′ = $v$, on suppose que la relation $(v-u)^2 = Ru$ est vérifiée. (1) Trouver l'enveloppe de Δ. Cette enveloppe est une courbe Γ, unicursale du sixième ordre et de la quatrième classe, possédant quatre points de rebroussement. En désignant pas $\phi$ l'angle O′O$x$, on démontrera que les coordonnées d'un point de cette courbe correspondent aux formules

$$x = R\cos\phi\,(1+\sin^2\phi), \quad y = -R\sin\phi\cos^2\phi.$$

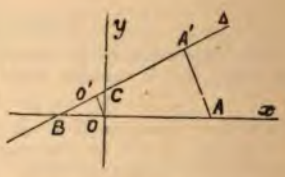
(2) Déduire de ces formules les propriétés essentielles de cette courbe.

---

*Solution by the* PROPOSER.

Ces propriétés sont les suivantes :—

(*a*) Une tangente à Γ rencontre OA en B ; la perpendiculaire élevée à cette tangente en B rencontre la perpendiculaire O$y$, à OA, en C ; BC est constant et égal à R.




En d'autres termes : la transformée orthotangentielle de Γ est une hypocycloïde à quatre rebroussements.

(*b*) Déduire de là, une construction par points et par tangentes de Γ.

(*c*) Construction du centre de courbure, en un point M de Γ.

(*d*) Démontrer que la développée de Γ est une hypocycloïde à quatre rebroussements.

(*e*) Rectification de Γ.

(*f*) Aire de Γ.

(2) Cette courbe remarquable Γ a été étudiée, par M. D'OCAGNE, dans un système particulier de coordonnées. (Voyez : *Nouvelles Annales*, 1888, p. 148 ; et, aussi, l'ouvrage.)— *Coordonnées Parallèles et Axiales*, par MAURICE D'OCAGNE. (Gauthier-Villars, 1888.)

---

**8553.** (Professor MAHENDRA NATH RAY, M.A., LL.B.)—If $x$, $y$, $z$ denote the distances of the orthocentre from the angular points of the triangle ABC, prove that

$$xyz\,(ax+by+cz)^3 + abc\,(x^2+y^2+z^2)(ax+by+cz)^2 - 4a^3b^3c^3 = 0.$$

---

*Solution by* R. KNOWLES, B.A.; Prof. SARKAR, *and others*.

$x$, $y$, $z$ are respectively equal to 2R cos A, 2R cos B, 2R cos C, and we obtain $xyz = 8R^3\cos A\,.\cos B\,.\cos C$, $(ax+by+cz) = abc/R$,

$$x^2+y^2+z^2 = 4R^2(\cos^2 A+\cos^2 B+\cos^2 C),$$

and the result in the question becomes

$$4a^3b^3c^3(\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C-1) = 0.$$

[For another solution, see Vol. XLVI., p. 60.]

Digitized by Google

**11259.** (The late Professor CLIFFORD, F.R.S.)—If a hexagon can be inscribed in a cubic curve so that the first, third, fifth sides pass through a fixed point $a$ on the curve, and the second, fourth, sixth pass through a fixed point $b$ on the curve, prove that, when the point $a$ is given, there are eight positions of $b$, which can be found by linear construction when the points of inflexion are known.

---

*Solution by* Professor NASH, M.A.

This depends upon the following theorem:—If the lines joining any point $a$ of a cubic to the nine inflexions, 123, &c., meet the curve again in the points (1), (2), &c., and each of these points be joined to the other eight inflexions, the 72 lines so obtained will meet the curve again in only eight points, one pair of points corresponding to each of the four sets of three lines through the inflexions. Also, if $b$, $c$ be one pair of these points, the three points $a$, $b$, $c$ are such that the line joining any two of them passes through the tangential of the third.

Let 123, 456, 789 be one set of three lines through the inflexions, and let (1, 2) denote the point where the line joining the point (1) to the inflexion 2 meets the curve again. The two sets of three lines $a2$ (2), (1) 2 (1, 2), 123, and $a$ (1) 1, 222, (2) (1, 2), form two cubics passing through eight points of the given cubic, and therefore meet it again in the same ninth point; hence the line (2)(1, 2) must pass through the inflexion 3, or in other words (2, 3) coincides with (1, 2). In the same way it can be shown that the points (3, 1), (4, 5), (5, 6), &c., coincide with (1, 2), and that the points (1, 3), (2, 1), (3, 2), (4, 6), &c., all coincide.

Hence, starting with each set of lines through the inflexions, we get one pair of points. I may remark that if the equation of the cubic be in the form $x^3+y^3+z^3+6mxyz=0$, and the coordinates of $a$ be (XYZ), the coordinates of the real points $b$, $c$ will be (YZX), (ZXY), and the coordinates of the imaginary points will be obtained by multiplying the coordinates of $a$, $b$, and $c$ by 1, $w$, $w^2$, or by 1, $w^2$, $w$ respectively, where $w^3=1$. The coordinates of the points (1), &c. are (XZY), (ZYX), (YXZ), the imaginary points being obtained from these in the same way. Some of the properties of these points are given in No. 1888, *Reprint*, V., p. 56.

Since $a1$ (1) are collinear, their tangentials are collinear, or the tangentials of $a$ and (1) are collinear with 1. Also, since $b2$ (1) are collinear, and also $c3$ (1); therefore, by the property of cubics through eight points, the line $bc$, and the line joining 1 to the tangential of (1) meet the curve in the same point, that is, $bc$ passes through the tangential of $a$, and similarly the lines joining $ca$ or $ab$ pass through the tangential of the remaining point.

Now let ABCDEF be the hexagon, the sides AB, CD, EF meeting in $a$, and the sides BC, DE, FA in $b$, and let $a'$ denote the tangential of $a$. The two sets of three lines BC$b$, $aaa'$, AD, and AB$a$, DC$a$, $ba'$ make up two cubics through eight points of the given cubic, and therefore meet it again in the same ninth point; hence AD and $ba'$ meet the cubic in the same point $c$. Similarly it may be shown that the lines BE, CF pass through the point $c$. (It, of course, follows from Brianchon's theorem that these lines meet in a point.) Now since the relation between $a$ and



Digitized by Google

$b$ is reciprocal, $ca$ must pass through $b'$, and therefore $ab$ through $c'$. Hence the three points are such that the line joining any two of them passes through the tangential of the third, and, any one of them being given, the other two are determined as above. If this relation is satisfied, the side AB of the hexagon may be taken arbitrarily. If AB is a tangent at A, or AB coincide, then DE coincide and also CF. If AB is the tangent at $a$, the vertices of the hexagon are the points $a$, $b$, $c$, and their tangentials.

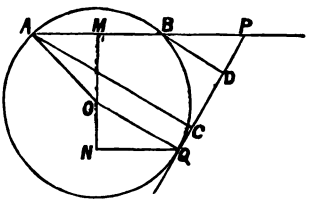
It is obvious that if the cubic is nodal there are only two positions of $b$ instead of eight, and the hexagon is impossible if the cubic is cuspidal.

---

**11304.** (The Editor.)—On the prolongation of a chord AB of a circle, find a point P such that, if AC, BD be perpendiculars on the tangent from P, the rectangle AC . BD may be given.

*Solution by* D. Biddle; M. Brierley; *and others.*

Let MN, drawn perpendicular to AB, be the side of a square which represents the given area. Draw NQ parallel to AB, to meet the circumference of the circle in Q. Join Q to O the centre, and draw the tangent QP cutting AB (produced) in P. Then P is the point required. For




$MN : PQ = AC : AP = BD : BP$,

$\therefore MN^2 : PQ^2 = AC . BD : AP . BP$.

But (Euc. III. 36), $AP . BP = PQ^2$; therefore $AC . BD = MN^2$.

---

**11172.** (Professor Brouilhet.)—Parmi tous les cônes de révolution de même volume, déterminer celui dont la surface latérale est minima.

*Solution by* H. W. Curjel, B.A.; Professor Zerr; *and others.*

Let the height $= x$, and the radius of base $= y$; then the volume is proportional to $xy^2$, and the lateral surface to $y(x^2+y^2)^{\frac{1}{2}}$. Therefore for a minimum value of lateral surface $u = y^2(x^2+y^2)$ is a minimum; therefore $du = 0$ and $d(xy^2) = 0$; therefore

$$(2yx^2+4y^3)\,dy + 2y^2x\,dx = 0 \quad \text{and} \quad 2xy\,dy + y^2dx = 0;$$

therefore $\dfrac{2x^2y+4y^3}{2xy} = \dfrac{2y^2x}{y^2}$; whence $\sqrt{2} \cdot y = x$;

therefore, if $\alpha$ is the semi-vertical angle, $\tan\alpha = 1/\sqrt{2}$.

Digitized by Google

**11351.** (G. Heppel, M.A.)—If $\alpha$, $\beta$, $\gamma$, $\delta$ are the interior angles made by the diagonal AC of a quadrilateral with the sides AB, BC, CD, DA, and $r$ and R the radii of the circles passing through the intersections of the internal and external bisectors of the angles of the quadrilateral, prove that

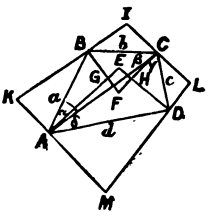
$$r = \tfrac{1}{4}AC\,\frac{\cos\frac{1}{2}(\alpha+\gamma)\cos\frac{1}{2}(\beta-\delta) \sim \cos\frac{1}{2}(\beta+\delta)\cos\frac{1}{2}(\alpha-\gamma)}{\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\gamma+\delta)\cos\frac{1}{2}(\alpha-\gamma)\cos\frac{1}{2}(\beta-\delta)},$$

$$R = \tfrac{1}{4}AC\,\frac{\cos\frac{1}{2}(\alpha+\gamma)\cos\frac{1}{2}(\beta-\delta) + \cos\frac{1}{2}(\beta+\delta)\cos\frac{1}{2}(\alpha-\gamma)}{\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\gamma+\delta)\cos\frac{1}{2}(\alpha-\gamma)\cos\frac{1}{2}(\beta-\delta)}.$$

---

*Solution by* Professors Zerr, Bhattacharya, *and others.*

Let EFGH be the quadrilateral formed by the internal bisectors, IKML the quadrilateral formed by the external bisectors; then




$$GE = \frac{d\sin\frac{1}{2}D\sin\frac{1}{2}(C+D) - a\sin\frac{1}{2}B\sin\frac{1}{2}(B+C)}{\sin\frac{1}{2}(B+C)\sin\frac{1}{2}(C+D)}$$

$$= \frac{AC}{2}\left\{\frac{\sin\gamma\cos\frac{1}{2}(\beta-\delta)\sin\frac{1}{2}(\alpha+\beta) - \sin\beta\cos\frac{1}{2}(\alpha-\gamma)\sin\frac{1}{2}(\delta+\gamma)}{\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\delta+\gamma) \times \cos\frac{1}{2}(\beta-\delta)\cos(\alpha-\gamma)}\right\}.$$

$$HE = \frac{AC}{2}\left\{\frac{\sin\gamma\sin\frac{1}{2}(\alpha+\delta)\cos\frac{1}{2}(\beta-\delta) - \sin\delta\sin\frac{1}{2}(\beta+\gamma)\cos\frac{1}{2}(\alpha-\gamma)}{\sin\frac{1}{2}(\delta+\gamma)\cos\frac{1}{2}(\delta+\gamma)\cos\frac{1}{2}(\beta-\delta)\cos\frac{1}{2}(\alpha-\gamma)}\right\};$$

$$IL = \frac{AC}{2}\left\{\frac{\sin\alpha\cos\frac{1}{2}(\delta+\gamma)\cos\frac{1}{2}(\beta-\gamma) + \sin\delta\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\gamma)}{\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\delta+\gamma)\cos\frac{1}{2}(\alpha-\gamma)\cos\frac{1}{2}(\beta-\delta)}\right\};$$

$$KI = \frac{AC}{2}\left\{\frac{\sin\beta\cos\frac{1}{2}(\alpha+\delta)\cos\frac{1}{2}(\alpha-\gamma) + \sin\alpha\cos\frac{1}{2}(\beta+\gamma)\cos\frac{1}{2}(\beta-\delta)}{\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\gamma)\cos\frac{1}{2}(\beta-\delta)}\right\};$$

$$r = \frac{GH}{2\sin\frac{1}{2}(B+C)} = \frac{GH}{2\cos\frac{1}{2}(\alpha-\gamma)};\quad R = \frac{KL}{2\cos\frac{1}{2}(\alpha-\gamma)}.$$

Substituting the values of GH, KL in terms of GE and HE, IL and KI, and reducing, the results follow.

---

**11297.** (Professor Chakravarti, M.A.)—The extremities of a uniform heavy beam, of length $2a$, slide on a smooth wire in the form of the curve $r = a(1-\cos\theta)$, the prime radius being vertical, and the vertex of the curve downwards. Prove that, if the beam be placed in a vertical position, and displaced with a velocity just sufficient to bring it into a horizontal position, and $\theta$ be the angle through which the rod has turned after a time $t$,

$$\tan\theta = \tfrac{1}{2}\left\{\epsilon^{(3g/2a)^{\frac{1}{2}}t} - \epsilon^{-(3g/2a)^{\frac{1}{2}}t}\right\}.$$

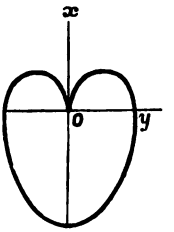
Digitized by Google

*Solution by* Rev. T. Galliers, M.A.; Prof. Radikanshuan; *and others.*

Taking axes as in the figure the equation of energy is $m(\dot{x}^2+\dot{y}^2+k^2\dot{\theta}^2) = C-2mgx$, which, on substituting

$$x = -a\cos^2\theta, \quad y = -a\sin\theta\cos\theta,$$

becomes $m(a^2+k^2)\dot{\theta}^2 = C+2ga\cos^2\theta$.



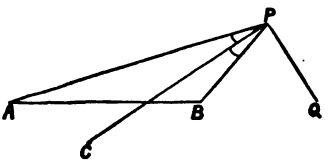

By the Question, $\dot{\theta} = 0$ when $\theta = \frac{1}{2}\pi$; therefore $C = 0$, also $k^2 = \frac{1}{2}a^2$; hence $\sec\theta\,.\,\dot{\theta} = (3g/2a)^{\frac{1}{2}}$ and $\log\cot(\frac{1}{4}\pi-\frac{1}{2}\theta) = (3g/2a)^{\frac{1}{2}}t$, whence the stated result follows.

---

**11316.** (G. E. Crawford, M.A.)—If A, B be two fixed points, and P a movable one such that one bisector of the angle APB passes through a fixed point, find (1) the locus of P; and (2) the envelope of the other bisector.

---

*Solution by* Professor Schoute; H. W. Curjel, B.A.; *and others.*




1. The locus of P is the locus of the common points of the corresponding circles of two pencils in projective relation (homographic pencils). As the circles with an infinite radius correspond to one another, the general result. *i.e.*, the bicircular quartic with C as a third node, degenerates in two parts, the line at infinity and a unicursal circular cubic.

The latter curve forms the locus in question; it passes through A, B and twice through C.

(For this result see Crelle's *Journal*, Vol. xlv., 375, and xcix., 98.)

2. The envelope of PQ is the negative pedal curve of the unicursal circular cubic with respect to the double point C, *i.e.*, a parabola, touching AB.

---

**5322.** (J. L. McKenzie, B.A.)—A triangle ABC is inscribed in a conic; through C a line is drawn parallel to the tangent at A, and meeting the conic in D; through A a line is drawn parallel to BC, and meeting the conic in E; prove that DE is parallel to AB.

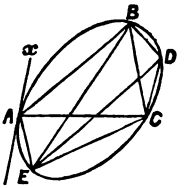
Digitized by Google

*Solution by* W. J. GREENSTREET, M.A.; Prof. SARKAR; *and others.*

DCBAE is a pentagon. Applying PASCAL'S theorem to this pentagon and tangent at A, we have the intersections of

DC and the tangent at A
BC and AE
AB and DE
} are collinear.

But DC and $Ax$, BC and AE, intersect on the line at infinity; therefore AB is parallel to DE.



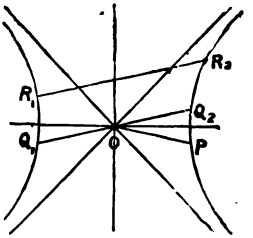

---

**11354.** (W. J. GREENSTREET, M.A.)—A, B, C, D are the points of intersection of a circle with an equilateral hyperbola. If AB passes through the centre of the hyperbola, then CD will pass through the centre of the circle.

---

*Solution by* Professor SCHOUTE; R. KNOWLES, B.A.; *and others.*

If OP and $Q_1OQ_2$ are two equal diameters of the hyperbola, the involution obtained by joining P to the extremities $Q_1Q_2$, $R_1R_2$ of chords parallel to $Q_1Q_2$, contains two right angles ($Q_1PQ_2$ and the parallels through P to the asymptotes); therefore it is generated by the rotation of a right angle about P; *i.e.*, the angle $R_1PR_2$ is right, and the circle described on $R_1R_2$ as diameter passes through the extremities of the diameter OP of the hyperbola.




---

**11299.** (Professor B. O. PIERCE.)—P is any point of a plane closed oval, QPQ′ a straight line drawn in the plane in a given constant direction so that QP, which is equal to PQ′, is a given finite continuous and single-valued function of the coordinates of P. Show that, as P moves around the curve, Q and Q′ will trace out two closed loops the sum of the areas of which is double the area of the given oval. In general, if $QP = \lambda PQ'$, $\lambda + 1$ times the area of the oval is equal to the area traced out by Q plus $\lambda$ times the area traced out by Q′. [*See* WOLSTENHOLME, Prob. 2400.]

---

*Solution by* H. W. CURJEL, B.A.; Prof. BHATTACHARYA; *and others.*

Let $P_1P_2$ be two points in which the given oval A is cut by a straight line drawn in the given constant direction, such that $P_1P_2$ lies entirely within the oval, and




Digitized by Google

let $Q_1Q_2$ be the corresponding points on the locus B of Q, and $Q_1'Q_2'$ be the corresponding points on the locus C of Q'.

Now $\quad Q_1P_1 = \lambda P_1Q_1', \quad$ and $\quad Q_2P_2 = \lambda P_2Q_2';$

therefore $\quad Q_1Q_2 + Q_2P_1 = \lambda\,(P_1P_2 + P_2Q_1'),$

$\quad Q_2P_1 + P_1P_2 = \lambda\,(P_2Q_1' + Q_1'Q_2');$

therefore $\quad P_1P_2 - Q_1Q_2 = \lambda\,(Q_1'Q_2' - P_1P_2);$

therefore $\quad (\lambda+1)\,P_1P_2 = \lambda Q_1'Q_2' + Q_1Q_2.$

Multiply each side by an infinitesimal quantity $dx$; we get the area of the parts of the curves between $Q_2'Q_1$ and a parallel to it at the distance $dx$.

Hence we see that $(\lambda+1) \times$ area of A $= \lambda \times$ area of C + area of B.

If $\lambda = 1$, area of B + area of C = 2 area of A.

---

**8509.** (Professor Sylvester.)—Show that the discriminant of the form $ax^5 + b\lambda\, x^4y + c\lambda^2x^3y^2 + c\mu^2x^2y^3 + b\mu xy^4 + ay^5$ will be a rational integral function of the quantities $a$, $b$, $c$, $\lambda\mu$, $\lambda^5 + \mu^5$, and of the second degree only in respect of the last of them.

---

*Solution by* H. J. Woodall.

By working in the usual manner, only that (in the later steps) the coefficients of powers and products of $\lambda$ and $\mu$ are changed for convenience, it can be shown, with some little difficulty, that the result is in the form

$$(\lambda^5+\mu^5)^2\left\{A\lambda^4\mu^4 + B\lambda^3\mu^3 + \ldots + E\right\}$$
$$+ (\lambda^5+\mu^5)\left\{A_1\lambda^7\mu^7 + B_1\lambda^6\mu^6 + C_1\lambda^5\mu^5 + \ldots + H_1\right\}$$
$$+ \left\{A_2\lambda^{10}\mu^{10} + B_2\lambda^9\mu^9 + C_2\lambda^8\mu^8 + \ldots + K_2\right\}.$$

---

**11241.** (J. J. Barniville.)—Evaluate or seriate

$$\sqrt{1+\sqrt{1+\sqrt{1+\&c.}}}, \quad \log 1 + \log\overline{1 + \log \&c}\ldots\ldots\ldots (1, 2).$$

---

*Solution by* Rev. T. R. Terry, M.A.; H. W. Curjel, B.A.; *and others.*

If $x$ and $y$ be the values of the expressions—(1) $x^2 = 1 + x$, therefore $x = \frac{1}{2}(\sqrt{5}+1) = 1{\cdot}618$; (2) $e^y = 1 + y$, therefore $y = 0$ is one root, and it may easily be proved that there is no other real root. For, if

$$z = e^y - (1+y), \text{ then } z_1 = e^y - 1,\ z_2 = e^y;$$

Digitized by Google

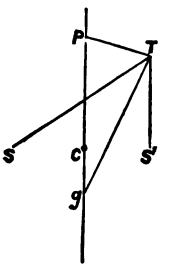
therefore $z$ is positive for all positive values of $y$. Putting $y = -Y$, we have $z = e^{-Y} - (1 - Y)$, $z_1 = -e^{-Y} + 1$, $z_2 = e^{-Y}$;
therefore $z$ is positive for all negative values of $y$.

---

**11334.** (Professor LAMPE, LL.D.)—Let P be a point of the transverse axis of a conic section whose focal axis is horizontal, T the point of contact of one of the tangents drawn from P to the curve. A material point glides without friction from P to T, the initial velocity being zero. If P be determined in such a way that the time of descent is a minimum, P proves to be a fixed point for a system of confocal conics, and the problem is easily reduced to a well known movement on the chords of a circle.

---

*Solution by* H. W. CURJEL, B.A.; Prof. DE WACHTER; *and others.*




Let normal at T meet the transverse axis in $g$, and let S and S′ be the foci, and C the centre. Then PTS′$g$S are concyclic and P$g$ is the vertical diameter of circle PTS′$g$S; therefore time of sliding down PT = time of sliding down P$g$, and is therefore a minimum when P$g$ is a minimum; *i.e.*, when P$g$ = SS′, since the circle always passes through SS′; therefore, for a minimum value of the time of sliding down PT, CP = CS, and P is therefore fixed for the system of conics whose foci are S, S′.

---

**10334.** (F. R. J. HERVEY.)—A pack of N = $mn$ cards is dealt into $n$ packs, which are collected in the order of dealing, placing the 1st on the 2nd, these on the 3rd, and so on; and the process repeated with the same or different values of $n$. Show that (1) deals with different values of $n$ are convertible in order; and hence (2) the cycles in which the cards are interchanged by any deal are independent of preceding deals. [For instance, 36 cards, in 4 similarly numbered suits, may be so arranged that, being dealt sometimes into 4, sometimes into 6, the first never disarranges the order of suits, nor the second the order of pips.] (3) Give the conditions which admit of any card being brought to a given place; and (4), this being possible, show, in the most general case, how to assign numbers $p$, $q$ ... whose product = N, so that it may not be necessary to use more than $p-1$ deals of one sort, $q-1$ of another, and so on.

---

*Solution by the* PROPOSER.

1. Let A, B denote the places of a card, counting from the top of the pack, before and after dealing respectively, and let $A = nx + y$. Then

Digitized by Google

$B = my - x$, and, eliminating $y$, $mA = x(mn+1) + B$, or $B \equiv mA$ for the modulus $N+1$.

2. Suppose the cards to be numbered according to their original places, and let the substitution S, already performed upon the *places*, become, by a new deal, TS. The corresponding operations performed upon the *cards* are respectively $S^{-1}$ and $(TS)^{-1} = (ST)^{-1} = T^{-1}S^{-1}$; thus the effect of the new deal is $T^{-1}$, independently of S. If dealing into $n$ packs replaces the card C by D, $D \equiv -nC \pmod{N+1}$. The numbers within the rectangle are the places of the cards in the instance given, the Roman numerals denoting the suits. The rows are cyclically interchanged by dealing into 6, and the columns by dealing into 4.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| I. | 1 | 9 | 7 | 26 | 12 | 34 | 10 | 16 | 33 |
| II. | 6 | 17 | 5 | 8 | 35 | 19 | 23 | 22 | 13 |
| III. | 36 | 28 | 30 | 11 | 25 | 3 | 27 | 21 | 4 |
| IV. | 31 | 20 | 32 | 29 | 2 | 18 | 14 | 15 | 24 |

3. Let $a, b, c, \ldots$ be the divisors of N, and let their powers give cycles of $\alpha, \beta, \gamma, \ldots$ remainders respectively. The conditions are that $N+1$ shall be prime, and that the products of powers of $a, b, c, \ldots$ shall give N different remainders, which requires that the L.C M. of $\alpha, \beta, \gamma, \ldots$ shall be N. This cannot be if N be divisible by 8, since every divisor is then the remainder of a square number. In every other case up to $N = 150$ (which is as far as I have gone), I find at least one primitive root of $N+1$ among the divisors of N.

4. Of the cycle numbers, let $\alpha, \beta, \gamma, \ldots$ be any set whose L.C.M. is N; make $p = \alpha$, and let $q, r, \ldots$ be the factors successively contributed to the L.C.M. by $\beta, \gamma, \ldots$. Now, $a^e b^f \equiv a^h b^k$ gives $b^{e-h)(f-k)} \equiv 1$, or $\beta$ divides $\alpha(f-k)$. Hence, by restricting $e$ and $f$ to be $< p$ and $q$ respectively, we get $pq$ distinct products $a^e b^f$. Similarly [since $(a^e b^f)^{pq} \equiv 1$], if $g$ be $< r$, we have $pqr$ distinct products $a^e b^f c^g$, and so on. For instance, let $N = 60$, and take the values 20, 3, 12, 6, of $m$, which give cycles of 5, 10, 15, 60; the 60 terms of the product

$$(1 + 20 + 20^2 + 20^3 + 20^4)(1 + 3)(1 + 12 + 12^2)(1 + 6),$$

are all different for the modulus 61, and a given change of place requires at most 8 deals.

---

**10766 & 10794.** (Professor MORLEY, M.A.)—10766. In an infinite mass of perfect fluid are three straight parallel vortex-filaments, of unequal strengths. Show that, if the cross section is once an equilateral triangle, it is always an equal equilateral triangle.

10794. In an infinite mass of perfect fluid are three straight parallel vortex-filaments of unequal strengths. Show that the circular cylinder which passes through the filaments is orthogonal to a fixed circular cylinder.

Digitized by Google

*Solution by* D. EDWARDES, B.A.

10766. If $\varpi m_1$, $\varpi m_2$, $\varpi m_3$ are the strength of the vortices, then, if they are at any instant at the corners of an equilateral triangle, it is at once seen that the system is at that instant rotating about the centre of the system (*i.e.*, the point which is the centre of gravity of masses proportional to $m_1, m_2, m_3$), with angular velocity $(m_1+m_2+m_3)/a^2$, when $a$ is a side of the triangle. Now the centre always remains at rest throughout the motion; hence at the next instant the vortices are at the corners of an equilateral triangle, and so on continually.

10794. If $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ be the coordinates of the vortices, referred to the centre as origin, and $r_1$, $r_2$, $r_3$ their distances from the centre, it is easily seen that throughout the motion $m_1r_1^2+m_2r_2^2+m_3r_3^2=C$, a constant. The equation of the circle, being written down in a determinant form and reduced by means of the equations

$$m_1x_1+m_2x_2+m_3x_3=0, \quad m_1y_1+m_2y_2+m_3y_3=0,$$

then takes the form $x^2+y^2+\alpha x+\beta y-C=0$,

which cuts orthogonally $x^2+y^2+C=0$.

---

**11368.** (Professor HAUGHTON, F.R.S.) — Two rectangular simple vibrations, one of which has double the periodic time of the other, act upon the same particle; find the path described by the particle.

---

*Solution by* H. J. WOODALL.

We have $x = A\cos mt$,

$y = B\cos(2mt+\alpha) = B\cos\alpha\,(2\cos^2 mt-1) - B\sin\alpha\,.\,2\sin mt\,.\cos mt$.

Substitute for $\cos mt$ and find $\sin mt$; eliminate $mt$, and we get

$$4x^4B^2\sin^2\alpha+\{B\cos\alpha\,(2x^2-A^2)-yA^2\}^2 = 4x^2A^2B^2\sin^2\alpha;$$

when $\alpha=0$, we get $B(2x^2-A^2)=yA^2$, a parabola.

---

**11283.** (H. W. PYDDOKE, B.A.)—If $a$ and $b$ are whole numbers taken at random, show that the chance that $a/b$ is in its lowest terms is $6/\pi^2$.

---

*Solution by the* PROPOSER.

The chances that (1) $a$ is divisible by some prime number $x$ is $1/x$; (2) $b$ is so divisible is also $1/x$; (3) $a$ and $b$ are both divisible by $x=1/x$; (4) none or only one is divisible, that is, that $a/b$ is in lowest terms as far as $x$ is concerned, is $1-1/x^2$. The chance that $a/b$ is in lowest terms



Digitized by Google

is the chance that the preceding is true of $a/b$ for every value of $x$ that is a prime number (except unity), *i.e.*,

$$(1-1/2^2)(1-1/3^2)(1-1/5^2)(1-1/7^2)\ldots,$$

and so on for every prime number.

But the reciprocal of this expression is known and can be easily shown to be equal to $1/1^2+1/2^2+1/3^2+\ldots$, the well known series for $\frac{1}{6}\pi^2$.

Therefore the chance that $a/b$ is in lowest terms is $6/\pi^2$.

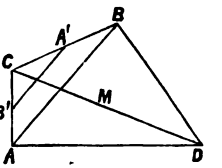
---

**11337.** (Professor CURTIS, M.A., S.J.)—At any point on a curve the radius of the circumcircle of the triangle formed by three tangents, when they become coincident, is $= \frac{1}{4}$ of the radius of curvature at the same point.

---

*Solution by* Professors SCHOUTE *and* SARKAR.

For the given curve that may touch the lines AC and BC in A and B, we substitute the parabola determined by these conditions. Of this parabola the join of the mid-points A′ and B′ of the sides BC and AC is a third tangent. Now the circumradius of triangle $A'B'C = \frac{1}{2}$ of circumradius CM of triangle $ABC = \frac{1}{4}$ of CD. By passing to the limit of coincidence of A and B the theorem is proved.




---

**9977.** (Professor LAMPE, LL.D.)—Investigate formulæ for the sums of the powers of the rational numbers, and, with the notation

$$S^k(x) = 1 + 2^k + \ldots + x^k,$$

prove, from general expressions, that

$$16S_1^5 = S_5 + 10S_7 + 5S_9, \quad 12S_2^3 = S_4 + 7S_6 + 4S_8, \quad 12S_1S_5 = -S_3 + 5S_5 + 8S_7,$$
$$30S_2S_4 = -S_3 + 15S_5 + 16S_7, \quad 72S_1S_2S_3 = 3S_4 + 42S_6 + 27S_8.$$

---

*Solution by* J. J. BARNIVILLE, M.A.

The sum of the general series can be expressed in terms of factorials, without the use of BERNOULLI's numbers, in the following manner:—

$$(1-x)^{-2-k}\left[1^k + \left\{2^k-(k+1)\,1^k\right\}x + \left\{3^k-(k+1)\,2^k + \frac{(k+1)\,k}{2}1^k\right\}x^2 + \ldots + \left\{2^k-(k+1)\,1^k\right\}x^{k-2} + x^{k-1}\right]$$
$$= 1^k + (1^k+2^k)\,x + (1^k+2^k+3^k)\,x^2 + \ldots \text{ (see Quest. 10388).}$$

Digitized by Google

Hence, after the Binomial theorem,

$$1^k + 2^k + \ldots + n^k = \frac{(n+k)!}{(k+1)!\,(n-1)!} \cdot \\ + \left\{2^k - (k+1)\,1^k\right\} \frac{(n+k-1)!}{(k+1)!\,(t-2)!} + \&c.,$$

where the factorials represent $k$ successive terms of the figurate series $1 \,.\, k+2$, $\frac{(k+2)(k+3)}{2}$, .... By this principle we can also sum

$$n \,.\, 1^k + (n-1)\,2^k + \ldots + 2\,(n-1)^k + 1 \,.\, n^k,$$

and in general

$$\frac{(n+r)!}{(n-1)!\,(r+1)!} \cdot 1^k + \frac{(n+r-1)!}{(n-2)!\,(r+1)!} \cdot 2^k + \ldots + (r+2)(n-1)^k + 1 \,.\, n^k.$$

[For other solutions, see p. 41 of Vol. LII.]

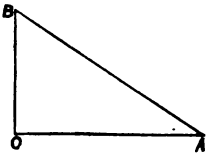
---

**11041.** (Professor MOREL.) — Etant donné un triangle rectangle isoscèle OAB, on demande : (1) l'équation générale des hyperboles équilatères circonscrites à ce triangle ; (2) si l'on considère la tangente à l'origine à chacune de ces hyperboles, on sait que cette droite est fixe. On suppose que sur cette droite on prenne un point fixe C. Sur ce point on mène la deuxième tangente à l'une quelconque des hyperboles précédentes. Lieu des points de contact de ces deuxièmes tangentes.—Ce lieu est composé d'une droite fixe et d'une conique. Si le point C se déplace sur la tangente fixe commune à toutes les hyperboles équilatères précédentes, on demande le lieu des centres des coniques précédentes.

---

*Solution by* R. KNOWLES, B.A. ; Professor CHAKRIVARTI ; *and others.*

Let OA = OB = $a$ ; the equation to the rectangular hyperbolas through OBA is




$$x^2 + kxy - y^2 - ax + ay = 0 \ldots\ldots (1),$$

where $k$ is variable ; the equation to the tangent at O is $x - y = 0$, and is therefore a fixed line ; the equation to a tangent at $x'y'$ is $(2x' + ky' - a)\,x + (kx' - 2y' + a)\,y - ax' + ay' = 0$ ; put $x = b = y$ the coordinates of C, then $k = (x-y)\,(a-2b)\,b\,(x+y)$. Substituting this value in (1), we have for the locus

$$x^2 - y^2 + xy\,(x-y)\,(a-2b)/b\,(x+y) - a\,(x-y) = 0 ;$$

which breaks up into the fixed line $x - y = 0$, and the conic

$$b\,(x^2 + y^2) + axy - ab\,(x+y) = 0 ;$$

if $xy$ be the coordinates of its centre $2bx + ay = ab$, and $ax + 2by = ab$ ; eliminating $b$, we get $2\,(x^2 - y^2) - a\,(x-y) = 0$ ; therefore the locus of the centres is the straight line $2\,(x+y) = a$, and it is at right angles to $x - y = 0$.

Digitized by Google

**10859.** (W. J. Greenstreet, M.A.)—$\phi(x)$ is a positive function of $x$, which does not increase between the values $a$ and $\infty$ of the variable; given $\int_a^\infty f(x)\,dx$ is finite, show that $\int_a^\infty f(x)\,\phi(x)\,dx$ is finite.

---

*Solution by* H. J. Woodall, Prof. Beyens; *and others.*

Expanding, we have

$$\therefore \int_a^\infty f(x)\,\phi(x)\,dx = \left[f(a)\,\phi(a) + f(x_1)\,\phi(x_1) + f(x_2)\,\phi(x_2) + \ldots\right] dx;$$

but $\phi(x)$ is $\not> \phi(a)$;

$\therefore$ this expansion is $\not> \phi(a)\left[f(a) + f(x_1) + f(x_2)\ldots\right] dx$;

*i.e.*, $\not> \phi(a)\int_a^\infty f(x)\,dx$, *i.e.*, is finite.

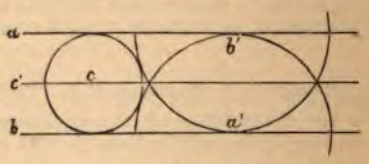
---

**11113.** (G. F. Howse.)—Three circles touch one another two and two; through each point of contact another circle is drawn touching the pair at that point and cutting the third at right angles. Show that these three circles are coaxal.

---

*Solution by* J. C. St. Clair; Professor Sarkar; *and others.*

Let A, B, C be the three circles, and C′ the circle having common contact with A, B, and cutting C orthogonally.




If we invert from the point common to A, B, C′, these circles become three parallel lines $a$, $b$, $c'$, (fig.), and C becomes a circle $c$ touching $a$, $b$, and having its centre on $c'$ (since $c$, $c'$ are orthogonal), which is therefore equidistant from $a$ and $b$. The circles A′, B become equal circles touching $b$, $c$ and $a$, $c$ respectively, and having their centres on $a$, $b$ respectively. It is evident from the original figure that A′, B′ must intersect; and it is also evident that the intersections of their inversions $a'$, $b'$ lie on $c'$. Hence A′, B′, C′ are c.p. coaxal circles.

---

**10894.** (Professor Bourrienne.)—Soient XOY un angle fixe et A un point fixe pris sur OX. On trace un cercle quelconque C tangent à OX et en un point D à OY; puis de A on mène à ce cercle une seconde tangente qui le touche en E. Démontrer que la droite DE passe par un point fixe.

Digitized by Google

*Solution by* GEORGE HEPPEL, M.A.

Take the bisector of XOY as axis of $x$, and let $\angle O = 2\alpha$, $OA = a$, $OD = b$, and let the tangent from A meet OY in F, where $OF = h$; then

$$2b = a + h - (a^2 + h^2 - 2ah\cos 2\alpha)^{\frac{1}{2}};$$

$$\therefore \quad h = \frac{b^2 - ab}{b - a\cos^2\alpha}.$$

Now the circle is $x^2 + y^2 - 2bx\sec\alpha + b^2 = 0$,

and F being $[h\cos\alpha\ h\sin\alpha]$, its polar DE is

$$hx\cos\alpha + hy\sin\alpha - b\sec\alpha\,(x + h\cos\alpha) + b^2 = 0.$$

This cuts the axis in $x = \dfrac{b^2 - bh}{b\sec\alpha - h\cos\alpha}$, and substituting the value of $h$, $x = a\cos\alpha$. [For another Solution, see Vol. LV., p. 77.]

---

**11022.** (G. HEPPEL, M.A.)—If A, B, C are the angles of a triangle, transform $(\sin B + \sin C)\sin B\sin C + (\cos C - \cos A)\cos A\sin C + (\cos B - \cos A)\cos A\sin B$ into an expression symmetrical with respect to A, B, C.

*Solution by the* PROPOSER; Professor MUKHOPÂDHYÂY; *and others.*

It is clear that $\cos\frac{1}{2}B\cos\frac{1}{2}C$ is a factor. Therefore, if the expression is symmetrical, $\cos\frac{1}{2}A$ is also a factor. Dividing by $\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$, the other factor is

$$8\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C\cos\tfrac{1}{2}(B-C) + \sec\tfrac{1}{2}A\cos A\left[4\sin\tfrac{1}{2}C\sin\tfrac{1}{2}(A-C) + 4\sin\tfrac{1}{2}B\sin\tfrac{1}{2}(A-B)\right],$$

and this ultimately reduces to

$$4\left[\cos A\cos B + \cos B\cos C + \cos C\cos A\right] - 2\left[\cos A + \cos B + \cos C\right] + 2.$$

---

**9299.** (A. GORDON.)—If a polygon of $n$ sides be inscribed in a circle, with vertices $A_1A_2A_3 \ldots A_n$ (taken in order), prove that the following conditions must be satisfied:—

$$(A_rA_s)(A_tA_1) + (A_1A_r)(A_sA_t) = (A_rA_t)(A_sA_1) \qquad \text{(EUC., Bk. VI.)}$$

$$(A_rA_s)(A_1A_t)(A_1A_u) + (A_sA_t)(A_1A_u)(A_1A_r) + (A_tA_u)(A_1A_r)(A_1A_s) = (A_rA_u)(A_1A_s)(A_1A_t),$$

$$A_rA_s(A_1A_t)(A_1A_u)(A_1A_v) + (A_sA_t)(A_1A_u)(A_1A_v)(A_1A_r) + (A_tA_u)(A_1A_r)(A_1A_s)(A_1A_v) + (A_uA_v)(A_1A_r)(A_1A_s)(A_1A_t) = (A_rA_v)(A_1A_s)(A_1A\ )(A_1A_u),$$

&c. &c.

Digitized by Google

Finally, $\{A_{12}$ the same as $(A_1A_2)$, &c.$\}$

$$(A_1A_3)\,A_{14}A_{15}A_{16}\ldots\ldots A_{1n} + (A_3A_4)\,A_{12}A_{15}A_{16}\ldots\ldots A_{1n} + \ldots\ldots$$
$$+ (A_{n-1}A_n)\,A_{12}A_{13}\ldots\ldots A_{1n-2} = (A_2A_n)\,A_{13}A_{14}\ldots\ldots A_{1n-1}$$

($r$, $s$, $t$, $u$, $v$ are supposed in order, but not necessarily consecutive.)

---

*Solution by* H. J. Woodall.

Since Euclid does not take count of sign, Book vi., quoted above, may be put

$$(A_rA_s)\,(A_1A_t) + (A_sA_t)\,(A_1A_r) = (A_rA_t)\,(A_1A_s) \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Multiply (1) by $(A_1A_u)$ and add $(A_tA_u)\,(A_1A_r)\,(A_1A\ )$ to both sides;

$$\therefore\ (A_rA_s)\,(A_1A_t)\,(A_1A_u) + (A_sA_t)\,(A_1A_u)\,(A_1A_r) + (A_tA_u)\,(A_1A_r)\,(A_1A_s)$$
$$= (A_rA_t)\,(A_1A_s)\,(A_1A_u) + (A_tA_u)\,(A_1A_r)\,(A_1A_s) = (A_1A_s)\,(A_rA_u)\,(A_1A_t)$$

by (1). Similarly from (3), (4), &c. Take the $(n-1)^{\text{th}}$ equation, and put $r=2$, $s=3$, $t=4$, &c. (*i.e.*, using all vertices), multiply by $A_1A_{n+1}$, *i.e.*, $A_{1.n+1}$ and add $(A_uA_{n+1})\,A_{1.2}A_{1.3}\ldots A_{1.(n-1)}$ we get (left-hand side)

$$= \{A_{13}A_{14}\ldots A_{1.(n-1)}\}\ \{(A_2A_n)\,A_{1.(n+1)} + A_{1.2}\,(A_nA_{n+1})\}$$
$$= \{A_{13}A_{14}\ldots A_{1.(n-1)}\}\ A_2A_{n+1}.A_{1.n} = (A_2A_{n+1})\,A_{13}A_{14}\ldots A_{1n};$$

*i.e.*, the theorem still holds.

---

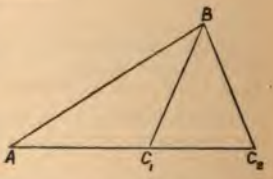
**8919.** (W. J. Greenstreet, M.A.)—In the ambiguous case, if A be given angle, $a$, $c$ given sides $(c>a)$, $C_1$, $C_2$ the two positions of the angular point opposite $c$, show that the radius of nine-point circle of $\Delta BC_1C_2$ is half that of nine-point circle of $\Delta ABC_2$ if $c=2a$, and is equal to radius of inscribed circle of $BC_1C_2$ if $\sin A = \frac{1}{4}\sqrt{3}$.

---

*Solution by* H. J. Woodall.

NP radius of $BC_1C_2 = \frac{1}{2}$ circum-radius $= a/4\sin C = r_1$;

NP radius of $BAC_2 = \frac{1}{2}$ circum-radius $= c/4\sin C = r_2$; therefore $r_1 : r_2 = a : c$.




In-radius of $BC_1C_2 = a\cos C\,.\tan\frac{1}{2}C$,

$$\frac{a}{\sin 4C} = a\cos C\tan\tfrac{1}{2}C,$$

if $\quad 1 = 4\sin C\,.\cos C\,.\tan\frac{1}{2}C$;

$$\therefore\ \cos\tfrac{1}{2}C = 8\sin^2\tfrac{1}{2}C\cos\tfrac{1}{2}C\,(2\cos^2\tfrac{1}{2}C-1),\quad \therefore\ 4\sin^2\tfrac{1}{2}C = 1,$$

$$\therefore\ \sin\tfrac{1}{2}C = \tfrac{1}{2}.\quad \therefore\ \sin C = \frac{\sqrt{3}}{2},\quad \sin A = \frac{a}{c}\sin C = \tfrac{1}{2}.\frac{\sqrt{3}}{2} = \tfrac{1}{4}\sqrt{3}.$$

Digitized by Google

**2041.** (The late Professor WOLSTENHOLME, D.Sc.)—The whole perimeter ($2a$) and one side ($c$) are given. Prove that the mean value of its area is $\frac{1}{8}\pi c\,[a\,(a-c)]^{\frac{1}{2}}$; and that the mean value of this mean value, $c$ being equally likely to have any value from 0 to $a$, is $\frac{\pi}{30}\,a^2$.

*Solution by* Profs. ZERR, RADIKANSHUAN, *and others.*

Since $c$, $x$ = the sides, A = area, and M = mean; then

$$A = \{a\,(a-c)(x-a)(a-x+c)\}^{\frac{1}{2}},$$

$$M = \int_a^{a+c} \{a\,(a-c)(x-a)(a-x+c)\}^{\frac{1}{2}}\,dx \Big/ \int_a^{a+c} dx = \tfrac{1}{8}\pi c\,[a\,(a-c)]^{\frac{1}{2}};$$

also the mean value of this mean is

$$M_1 = \tfrac{1}{8}\pi \int_0^a c\,[a\,(a-c)]^{\frac{1}{2}}\,dc \Big/ \int_0^a dc = \frac{\pi a^2}{30}.$$

**11191.** (Rev. T. ROACH, M.A.)—In an ellipse PQ is drawn perpendicular to SP, and $PS_1$, PQ meet CD in E, Q; prove that the radius of the circle about PEQ varies inversely as FE.

*Solution by* Rev. T. R. TERRY, M.A.; D. EDWARDS, B.A.; *and others.*

By Euclid vi. c., diameter of circle = PE . PQ/PF. If PS meets CD in E′, then EF = E′F, and PE = PE′ = AC, and, by similar triangles, PQ : PF = PE′ : E′F; therefore diameter of circle = $AC^2/EF$.

**11018.** (D. BIDDLE.)—In reference to reciprocals, prove that

$$\Sigma_0^\infty\,(1-a/A)^n = A/a, \quad \text{where } A > \tfrac{1}{2}a.$$

*Solution by* H. J. WOODALL; Professor BEYENS; *and others.*

$$\Sigma_0^\infty\,(1-a/A)^n = 1/\{1-(1-a/A)\} = A/a.$$

Convergent if $1-a/A$ is $< \pm 1$;

*i.e.*, $+1 > 1-a/A > -1$; $\therefore\ 0 > -a/A > -2$; $\therefore\ A > \frac{1}{2}a$.

**11232.** (The EDITOR.)—Prove that if, in a triangle ABC, the angles be such that $\Sigma\,(\cos A) = \sqrt{2}$, the circumcentre of the triangle will be on the circumference of the incircle.

Digitized by Google

*Solution by* Rev. T. R. Terry, M.A.; J. M. Stoops, B.A.; *and others.*

Since $\Sigma(\cos A) = \sqrt{2}$, therefore $4 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C = \sqrt{2} - 1$; therefore $R + r = R\sqrt{2}$, where $R^2 - 2Rr = r^2$, or $SI = r$.

---

**1850.** (Professor Sylvester.)—In a parabolic segment cut off by a line perpendicular to the axis two points are taken; find the mean value of such of the chords drawn through them as cut the segment in two points.

*Solution by* H. J. Woodall.

Take a point $(h, k)$ and draw chords in all possible directions.

This represents the variation of the inclination; then, to obtain mean values of length of chord for a position, say $(h, k)$, of this point. The number of positions of second point on any chord of the series varies as the length of the chord.

Hence mean length $= \Sigma(\text{length})^2 / \Sigma(\text{length})$ ..... (A).

Let $x - h = M(y - k)$ be any one chord; $x = b$ is perpendicular line; M varies from $(b-h)/[2(ab)^{\frac{1}{2}} - k]$ through 0 to $(b-h)/[-2(ab)^{\frac{1}{2}} - k]$. Let $(n_1)$, $(n_2)$ be extremities of chord, therefore $M(2an - k) = an^2 - h$ is the relation giving $n_1$ and $n_2$, the roots; therefore $n_1 - n_2 = \left(M^2 - \frac{Mk - h}{a}\right)^{\frac{1}{2}}$.

Length of chord $= [(an_1^2 - an_2^2)^2 + 4a^2(n_1 - n_2)^2]^{\frac{1}{2}} = a(n_1 - n_2)(4 + 4M^2)^{\frac{1}{2}}$

$$= 2a(n_1 - n_2)(1 + M^2)^{\frac{1}{2}} = 2a\left(M^2 - \frac{Mk - h}{a}\right)^{\frac{1}{2}}(1 + M^2)^{\frac{1}{2}},$$

and the variation is according to M in the integration and the limits as above. Result is fraction (B).

Next vary $k$, and thus the first point moves on the line $x = h$; then the number of possible positions of second point varies as area of parabola minus a triangle whose base is the chord $x = b$ and whose apex is at the point $(h, k)$; but all these triangles are equal ($k$ being varied); hence integrate both numerator and denominator of fraction (B) according to $k$ $(y)$, the limits being $\pm (4ah)^{\frac{1}{2}}$. Result is fraction (C).

Next in fraction (C). Area which contains possible positions of second point, when first point is $(h, y)$, is

parabola − above triangle $= \frac{4}{3}(ab^3)^{\frac{1}{2}} - 2(ab)^{\frac{1}{2}}(b - h) = 2(ab)^{\frac{1}{2}}(\frac{1}{3}b + h)$.

Therefore number of points for each value of

$h = 4(ah)^{\frac{1}{2}} \times 2(ab)^{\frac{1}{2}}(\frac{1}{3}b + h) = 8a(bh)^{\frac{1}{2}}(\frac{1}{3}b + h)$, limits $+b$ to 0;

therefore average length

$$= \frac{\iiint 2a\left(M^2 - \frac{My - x}{a}\right)(1 + M^2)\sqrt{x}\,(b + 3x)\,dM \,.\, dy \,.\, dx}{\iiint \left[\left(M^2 - \frac{My - x}{a}\right)\left(1 + M^2\right)\right]^{\frac{1}{2}}\sqrt{x}\,(b + 3x)\,dM \,.\, dy \,.\, dx},$$

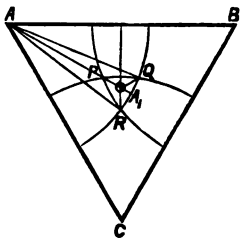
the limits being, for M, $y$, $x$, $(b-x)/[2(ab)^{\frac{1}{2}} - y]$ to $(b-x)/[-2(ab)^{\frac{1}{2}} - y]$, $\pm 2(ax)^{\frac{1}{2}}$, and 0 to $b$.

Digitized by Google

**2885.** (Artemas Martin, LL.D.)—Three horses are tethered in an equilateral triangular lot, each being fastened to a corner of the lot by a rope $r$ feet long, and there are, in the centre of the lot, $a$ square feet common to all three of the horses. Required the dimensions of the lot.

*Solution by* H. J. Woodall.

In the diagram, PQR is the common part.




$$AQ = AA_1 = \&c. = r.$$

Let $AB = x$, $AO = R = \frac{1}{3}\sqrt{3}\,.\,x$,

$$QOA_1 = 60°,\ AOQ = 120°.$$

$$\sin AQO = R \sin 120°/r = \sqrt{3}\,R/2r,$$

$$\cos AQO = (1 - 3R^2/4r^2)^{\frac{1}{2}},$$

$$\sin QAA_1 = \sqrt{3}\,\{(4r^2 - 3R^2)^{\frac{1}{2}} - R\}/4r;$$

whence we may find area QOA and $A_1AQ$.

Common area

$$PQR = 6\{A_1AQ - OAQ\},$$

$$a = 3r^2 \arcsin\left[\sqrt{3}\,\{(4r^2 - 3R^2)^{\frac{1}{2}} - R\}/4r\right] - \frac{3\sqrt{3}}{4} R\,\{(4r^2 - 3R^2)^{\frac{1}{2}} - R\}.$$

And then substitute for R; given $a$ and $r$ we must find R (or $x$) by approximation.

---

**5136.** (Professor Hendricks, M.A.)—If four bricks are placed on each other at random, with their longest axes horizontal and in the same vertical plane, show that the probability that the pile will stand is $\frac{2080}{3504}$.

*Solution by* Professor Schoute.

If $2l$ represents the length of the bricks, and the variables $x$, $y$, $z$, indicated in Fig. 1, are taken with the positive sign when the upper brick is deponed to the right with reference to the lower one, the conditions of equilibrium are

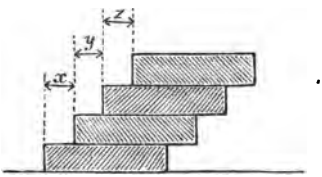


Fig. 1.

$$x^2 < l^2,\quad y^2 < l^2,\quad (x + \tfrac{1}{2}y)^2 < l^2,$$

$$z^2 < l^2,\quad (y + \tfrac{1}{2}z)^2 < l^2,$$

$$\left[x + \tfrac{1}{2}y + \tfrac{1}{2}z)\right]^2 < l^2;$$

whilst the possible values of $x$, $y$, $z$ are included in the condition $u^2 < 4l^2$, where $u$ stands for $x$, $y$, $z$.

Now let $x$, $y$, $z$ be the rectangular coordinates of a point in space. Then the points corresponding to positions of equilibrium will lie within the cube the faces of which are parallel to the planes of coordinates in a distance $l$, and between the three pairs

$$2x + y = \pm 2l,\quad 2y + z = \pm 2l,\quad 3x + 2y + z = \pm 3l$$



Digitized by Google

of parallel planes. In Fig. 2, that cube and those planes have been

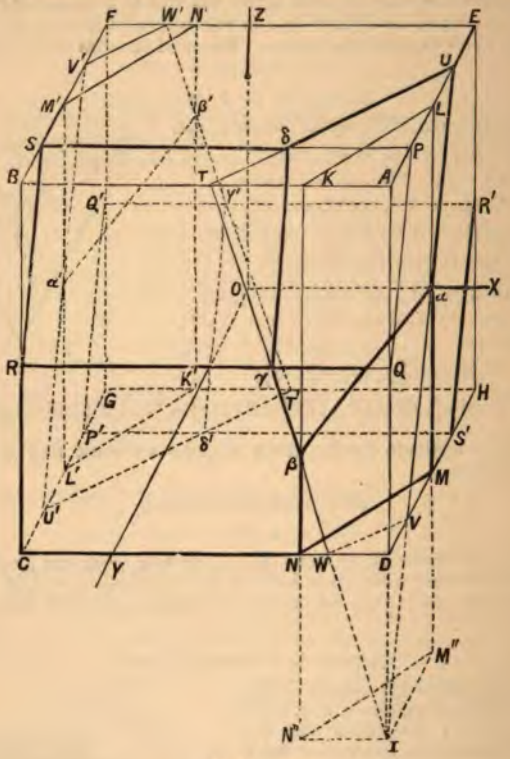



Fig. 2.

represented. In the followed order the latter are

$$\left.\begin{matrix} \text{KLMN} \\ \text{K'L'M'N'} \end{matrix}\right\}, \quad \left.\begin{matrix} \text{PQRS} \\ \text{P'Q'R'S'} \end{matrix}\right\}, \quad \left.\begin{matrix} \text{TUVW} \\ \text{T'U'V'W'} \end{matrix}\right\}.$$

Now we find for the portions of the cube taken away by the planes

$$2\,(\text{AITU} - \text{DIWV}) + 2\text{NMVW}\alpha\beta + 2\text{BRST}\gamma\delta,$$

or
$$2\,(\text{AITU}) + 2\,(\text{N''M''I}\alpha\beta - \text{N''M''INMD}) + 2\text{BRST}\gamma\delta,$$

*i.e.*,
$$\frac{54+3+22}{36}l^3 = 2\tfrac{7}{36}l^3.$$

So the probability in question is $\frac{8-2\frac{7}{36}}{64}$, or $\frac{209}{2304}$.

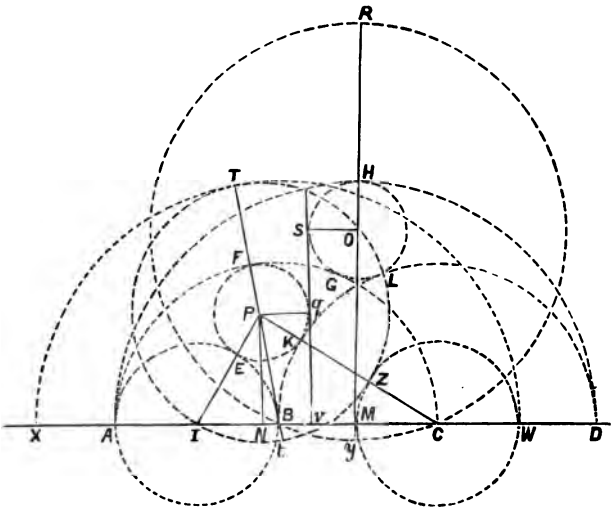
Digitized by Google

**11284.** (E. M. Langley, M.A.)—If B, C are points of trisection of a straight line ABCD, and AEB, AFGC, AHD, BKLD semicircles on AB, AC, AD, BD, show geometrically that, if circles EFK, GHL be drawn to touch them, they are equal and have a common tangent perpendicular to AD.

---

*Solution by* I. Arnold; James Rice; *and others.*

On AB draw the circle AEB, with centre I; and from B, with radius BA, draw the circle AFGC; with C as centre and AI as radius draw the circle MZW; and with M as centre and MA as radius draw the circle AHD. Draw MH perpendicular to AD, and make HR = AB = $2r$,




when $r$ is radius of smaller circle AEB. Through the points R, B, C describe the circle RBC, and let O be the centre; then is O the centre of the circle touching AHD, AFGC and BLD. And $r$ being the radius of the smaller circle, RM = $5r$, and BM, MC each = $r$; therefore MY = $\frac{1}{5}r$; consequently RY = $5r + \frac{1}{5}r = \frac{26}{5}r$; and OR or OY, $\frac{13}{5}r$; therefore OH or OS = $\frac{13}{5}r - 2r = \frac{3}{5}r$. Draw S$v$ perpendicular to AD. then $v$M = $\frac{3}{5}r$, and S$v$ is tangent to SHL. With B as centre and BW = $3r$ as radius draw the circle WTX. and through the point I draw the circle IZLT, touching the circle MZW in Z, and the circle WTX in T, and let P be the centre of this circle; it is also the centre of the circle EKF touching the two equal circles AFC, BLD, and AEB. Join PC and PI, and TF, PB which are in the same straight line. Now PC = $2r$ + PK, and PI = $r$ + PE (PK). Make $v$N = $v$M, then is MN = $\frac{6}{5}r$, and

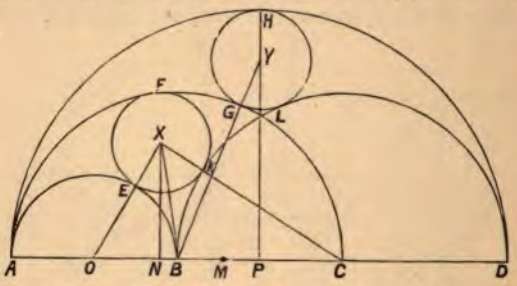
$$\text{IN} = 2r - \tfrac{6}{5}r = \tfrac{4}{5}r, \quad \text{and} \quad \text{NC} = \tfrac{14}{5}r.$$

Now assuming PK = P$q$ = N$v$, if the difference of the squares of IN,

Digitized by Google

NC be equal to the difference of the squares of PI, PC, then is PN perpendicular to IC and $Pq = PK = Nv = vM = OS$. On this assumption $CP = 2r + \frac{2}{5}r = \frac{12}{5}r$ and $PI = r + \frac{3}{5}r = \frac{8}{5}r$, and the difference of the squares is $\frac{144}{25}r^2 - \frac{64}{25}r^2 = \frac{80}{25}r^2$; but $NC = \frac{11}{5}r$ and $IN = \frac{4}{5}r$, and the difference of their squares is $\frac{80}{25}r^2$: hence PN is perpendicular to IC, and the radii $Pq$, OS of the circles are equal, and $Sqv$, the common tangent, is perpendicular to AD; therefore, &c.

[Mr. Rice proves the theorem thus:—O, X, Y are the centres of circles AEB, EFK, GHL. HYP is perpendicular to AB, and P is centre of the circle AHD. Let $r$ = radius of EFK, $\rho$ = radius of GHL,




R = radius of AEB: then $2OX^2 + CX^2 = 6OB^2 + 3BX^2$; therefore

$2(R+r)^2 + (2R+r)^2 = 6R^2 + 3(2R-r)^2$; therefore $r = \frac{3}{5}R$.

Again, $BP^2 + PY^2 = BY^2$; therefore $R^2 + (3R-\rho)^2 = (2R+\rho)^2$;

therefore $\rho = \frac{3}{5}R$, and $r = \rho$.

Draw XN perpendicular to AD, and let M be the mid-point of OC, and therefore also of BP; then we have

$2OC \cdot MN = CX^2 - OX^2$; $\therefore$ $MN = \frac{21}{30}R$; $\therefore$ $PN = \frac{6}{5}R = 2r$.

Hence the second part of the theorem is easily proved.]

---

**11363.** (William Hoover.)—A man runs a race, starting with velocity $v$, the direction of the wind appearing to him to make an angle $\alpha$ with his course. It then appears uniformly to veer round through an angle $\alpha$ during the race. The wind blows uniformly throughout and at right angles to his course. Show that, if $t$ be the time of the race, the length of the course is $vt\frac{\tan\alpha}{\alpha}\log(2\cos\alpha)$.

---

*Solution by* A. S. Ramsey, B.A.; H. W. Curjel, B.A.; *and others.*

If $x$ denote the distance traversed in time $t$, $\theta$ the angle through which the wind has veered, and V the absolute velocity of the wind; then

Digitized by Google

$V = \dot{x} \tan(\alpha + \theta)$, also initially $V = v \tan \alpha$; again $\theta \propto t = C \cdot t$ say, hence $\alpha = C \cdot t_1$, if $t_1$ is whole time. Therefore $\theta = \alpha t/t_1$. Again

$$\dot{x} = \frac{v \tan \alpha}{\tan(\alpha + \theta)}, \quad x = \int \frac{v \tan \alpha \cdot dt}{\tan(\alpha + \theta)} = v t_1 \frac{\tan \alpha}{\alpha} \int_0^{\alpha} \frac{d\theta}{\tan(\alpha + \theta)}$$

$$= v t_1 \frac{\tan \alpha}{\alpha} \Big[ \log \sin(\alpha + \theta) \Big]_0^{\alpha} = v t_1 \frac{\tan \alpha}{\alpha} \log(2 \cos \alpha).$$

---

**5368.** (H. W. Harris, B.A.)—Find the locus of the middle point of a chord of an ellipse, making any given angle with the tangent at its extremity; and point out some singularities of this curve.

---

*Solution by* Professor Schoute.

If the direction of the diameter OM (Fig. 1) is given, we find two points P and P′ of the ellipse, where the tangents make the given angle $\alpha$ with the chords parallel to the conjugate diameter. This proves that any diameter of the ellipse contains two points M and M′ of the locus, that differ from O. So O will be a centre of the locus. We seek the order of it by determining how often it passes through O. To that end we examine the envelope of the chord PQ (Fig. 2), that forms in a determinate sense the angle $\alpha$ with the diameter PO of one of its ends. This envelope is the first negative pedal of the locus of the foot Q of the normal OQ on PQ; *i.e.*, of the ellipse obtained by a rotation of the given ellipse through an angle $90° - \alpha$ about O and a multiplication of the radii vectores issuing from O by $\sin \alpha$. As the first negative pedal of this new ellipse with respect to O is a curve of the fourth class, it happens four times that the chord PM (Fig. 1), corresponding to a diameter OM, passes through O. So the locus in question passes eight times through O, and is of the tenth order.

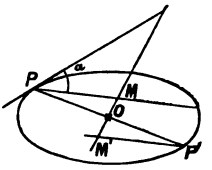


Fig. 1.

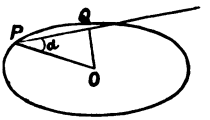


Fig. 2.

The locus $C^{10}$ must be unicursal. For its points M correspond one to one with the points P of the given ellipse. So the octuple point in O must count for thirty-six double points. This is really the case. For the eight branches passing through O consist of four pairs, each pair of which is formed by two inflexional branches with common inflexional tangent. So the ordinary number, twenty-eight, of double points represented by the octuple points is to be augmented with four times two.

[When, in the equation of the locus with respect to the axes of the given ellipse, we substitute $x'$ and $y'$ for $x^2$ and $y^2$, we obtain a new curve of the fifth order with a quadruple point in O.]

Digitized by Google

**9755.** (Rev. T. C. Simmons, M.A.)—If the base and vertical angle A of a triangle are given, show that the equation to the locus of the positive Brocard-point is $r^4 - 4Rr^3 \cos A \sin\theta + 4r^2R^2(3\sin^2\theta - \sin^2 A) - 16rR^3 \cos A \sin^3\theta + 16R^4 \sin^4\theta = 0$, where B is taken for origin, BC for initial line, and R for the given circum-radius.

---

*Solution by* Professor G. B. M. Zerr.

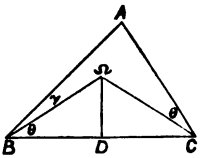
Let $B\Omega = r$, $\angle \Omega BD = \theta$; then

$\angle \Omega CD = C - \theta$, $\Omega D = r\sin\theta$, $DC = a - r\cos\theta$,

$r = 2R\sin\theta\,(C/b)$

$= 2R\sin\theta\{\sin A\cot B + \cos A\}$ .....(1),

$\cot B = (1 - \cot A\cot C)/(\cot A + \cot C)$.




From triangle ΩDC,

$$\cot C = \frac{a\cos\theta - r}{a\sin\theta}, \quad \cot B = \frac{r\cos A - a\cos(A+\theta)}{a\sin(A+\theta) - r\sin A}.$$

This value of cot B, substituted in (1), gives

$$ra\sin(A+\theta) = r^2\sin A + 2Ra\sin^2\theta,$$

therefore $2Rr\sin A\cos\theta = r^2 + 4R^2\sin^2\theta - 2Rr\cos A\sin\theta$;

squaring this last, writing $1-\sin^2\theta$ for $\cos^2\theta$, reducing and transposing, we get

$$r^4 - 4r^3 R\cos A\sin\theta + 4R^2r^2(3\sin^2\theta - \sin^2 A) - 16rR^3\cos A\sin^3\theta + 16R^4\sin^4\theta = 0.$$

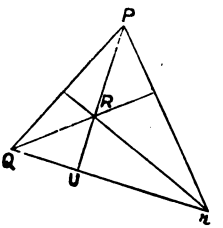
---

**4015.** (Dr. C. Taylor.)—Prove that any chord of a rectangular hyperbola subtends equal or supplementary angles at the ends of a perpendicular chord.

---

*Solution by* H. J. Woodall.

The four extremities of the chords mentioned are the vertices and orthocentre of some triangle.




Case (1) $\angle PQR$ = complement of $\angle QPr$

$= \angle PrR$;

case (2) $\angle PRQ = PUQ + RQU$,

and $\angle PrQ = PUQ - rPU$.

The triangle RQU is similar to $PUr$;

therefore $RQU = rPU$,

and $PRQ + PrQ = 2$ right angles.

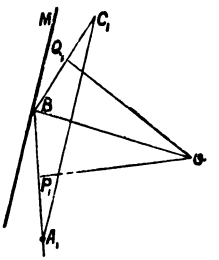
Digitized by Google

**4964.** (Rev. W. A. WHITWORTH, M.A.)—A regular "screw-polygon" is formed of equal straight lines. Any two adjacent sides contain an angle $\pi - \theta$, and the planes containing any side and either of the adjacent sides contain an angle $\psi$. Show that, if there be $n$ sides in a convolution,

$$\cos \pi/n = \cos \tfrac{1}{2}\theta \cos \tfrac{1}{2}\psi.$$

*Solution by* H. J. WOODALL.

Figure is the projection of two sides (AB, BC), and five other lines on the plane of the paper. Plane ABC contains AC, and $CC_1 = A_1A$; therefore mid-point of AC is in plane of paper, and therefore BO is in plane ABC, *i.e.*, BO is horizontal line. From Q draw QO′ horizontal, join QO, QO″ in plane BCD; therefore $OQO_1 = \frac{1}{2}\psi$ (because OQ, $O_1Q$ are perpendicular to intersection),




$$BOQ_1 = \pi/n, \quad QBM = \tfrac{1}{2}\theta = QOB.$$

$$\cos \pi/n = Q_1O/BO = Q_1O/QO \times QO/BO$$
$$= \cos \tfrac{1}{2}\psi \times \cos \tfrac{1}{2}\theta.$$

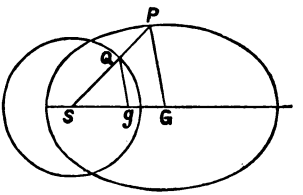
---

**11254.** (Professor TISSOT.)—Une circonférence ayant pour centre un foyer d'une conique, si l'on mène, à chaque normale de cette dernière courbe, une parallèle par le point où le rayon vecteur du pied de la normale rencontre la circonférence, les droites ainsi obtenues seront concourantes.

---

*Solution by* H. W. CURJEL, B.A.; Prof. RADHAKINSHAWA; *and others.*

Let a radius vector through the focus S cut the circle in Q and the conic in P. Let the normal at P meet the axis in G, and the parallel to PG through Q meet the axis in $g$.




$$SQ : Sg = SP : SG = 1 : e.$$

But SQ is constant; hence $Sg$ is constant, $g$ is fixed, and all the lines such as $Qg$ are concurrent.

---

**4609.** (Professor CHRISTINE LADD, B.Sc.)—A circle of radius R is completely surrounded by $n$ circles. These are in turn surrounded by successive rows of circles, each circle touching the two adjacent circles of its own row, and also two adjacent circles of the next inner row. Required the radius of the circles of the $m^{\text{th}}$ row.

Digitized by Google

*Solution by* H. J. WOODALL.

Let $\alpha = \pi/n$; then

$$(A_mA_{m+1})^2 = (A_mQ_m)^2 + (Q_mA_{m+1})^2;$$

$$OQ_m = r_m \cot\alpha:$$

$$OA_{m+1} = r_{m+1}\operatorname{cosec}\alpha;$$

$$(r_m + r_{m+1})^2 = r_m^2 + (r_{m+1}\operatorname{cosec}\alpha - r_m\cot\alpha)^2;$$

therefore

$$(r_m^2 + r_{m+1}^2)\cot^2\alpha - 2r_mr_{m+1}(\cot\alpha\operatorname{cosec}\alpha + 1) = 0;$$

therefore

$$r_{m+1}^2 - 2r_{m+1}r_m(\tan^2\alpha + \sec\alpha) + r_m^2 = 0;$$

$$r_{m+1}/r_m = \tan^2\alpha + \sec\alpha \mp \left[(\tan^2\alpha + \sec\alpha)^2 - 1\right]^{\frac{1}{2}}$$

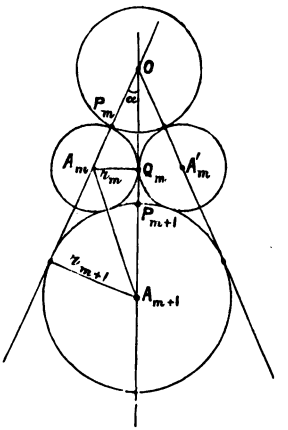
$$= \tan^2\alpha + \sec\alpha + \left[\tan^4\alpha + 2\tan^2\alpha\sec\alpha + \sec^2\alpha - 1\right]^{\frac{1}{2}}$$

$$= \tan^2\alpha + \sec\alpha + \tan\alpha\left[\sec^2\alpha + 2\sec\alpha\right]^{\frac{1}{2}};$$

$$(r_1 + R)\sin\alpha = r_1;$$

therefore $r_1 = R\sin\alpha/(1-\sin\alpha)$;

$$\therefore\ r_m = \left\{\tan^2\alpha + \sec\alpha + \tan\alpha\left[\sec^2\alpha + 2\sec\alpha\right]^{\frac{1}{2}}\right\}^{m-1} \times R\sin\alpha/(1-\sin\alpha).$$




---

**11355.** (A. J. PRESSLAND.)—If DEF be a transversal of the triangle ABC, cutting BC in D, CA in E and AB in F, prove that

$$\frac{AE \cdot AF}{CA \cdot AB} + \frac{AF \cdot BD}{AB \cdot BC} + \frac{CD \cdot AE}{BC \cdot CA} = 0.$$

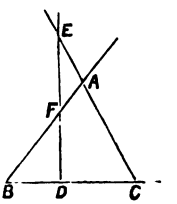
---

*Solutions by the* PROPOSER; H. J. WOODALL; *and others.*

(1) Only one parabola can be drawn to touch four given straight lines. It may be considered as the envelope of a line DEF cutting off segments AF and DC from the axes of $y$ and $x$, so that AF : DC :: $l$ : $m$. Its equation will be

$$(lx + my)^2 + 2(cm - al)(lx - my) + (cm - al)^2 = 0,$$

which touches CB at a distance $\frac{al - mc}{l}$ from B.




It may also be considered as the envelope of DEF, a line cutting off segments AE and BD from the axes of $y$ and $x$, so that AE : BD = $\mu$ : $\lambda$.

Digitized by Google

Its equation $(x\mu - y\lambda)^2 - 2(b\lambda + a\mu)(x\mu + y\lambda) + (b\lambda + a\mu)^2 = 0$,
which touches CB at a distance $\frac{a\mu + b\lambda}{\mu}$ from C; hence

$$\frac{a\mu + b\lambda}{\mu} - a = \frac{al - mc}{l} \quad \text{or} \quad \frac{\lambda l}{ac} = \frac{\mu l}{bc} - \frac{m\mu}{ab},$$

which is the required property.

In absolute magnitude

$$\begin{aligned} BD \cdot AF \cdot CA &= BD \cdot AF \cdot CE - AE \cdot AF \cdot BD \\ &= AE \cdot CD \cdot BF - AE \cdot AF \cdot BD, \\ CD \cdot AE \cdot AB &= AE \cdot CD \cdot BF + AE \cdot CD \cdot AF; \end{aligned}$$

subtracting, $BD \cdot AF \cdot CA - CD \cdot AE \cdot AB = -AE \cdot AF \cdot BC$

With the convention that AB, BC and CA are positive directions, this becomes

$$\frac{AE \cdot AF}{CA \cdot AB} + \frac{AF \cdot BD}{AB \cdot BC} + \frac{CD \cdot AE}{BC \cdot CA} = 0.$$

---

**5258.** (The late T. Cotterill, M.A.)—If a variable conic through four given points $a$, $b$, $c$, $d$, cut two given lines through $d$ in $p$ and $q$, prove that the six centres of perspective of the pairs of points $p$, $q$, and the three pairs of points taken from the points, $a$ $b$, $c$, describe straight lines.

*Solution by* Professor Schoute.

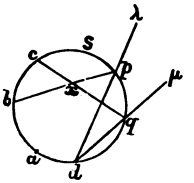


Fig. 1.

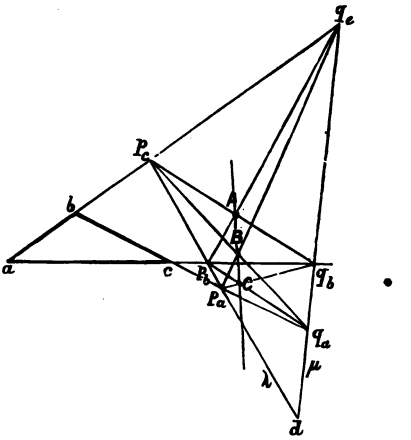


Fig. 2.

When S (Fig. 1) describes the pencil of conics through $a$, $b$, $c$, $d$, the



Digitized by Google

points $p$ and $q$ describe two ranges of points on the lines $\lambda$ and $\mu$, that are in projective relation with the pencil of conics and therefore with one another.

This proves that the two pencils of rays $bp$ and $cq$ also are projective. Thus the point $x$, common to $bp$ and $cq$, describes a conic through $b$ and $c$. But the two projective pencils $bp$ and $cq$ are in perspective too. For the common ray $bc$ corresponds to itself. Thus the conic described by $x$ breaks up in an improper part, the line BC, and another line; this line is the locus of $x$.

The six lines are represented in Fig. 2. If the sides $bc$, $ca$, $ab$ of triangle $abc$ intersect $\lambda$ and $\mu$ in $(p_a, q_a)$, $(p_b, q_b)$, $(p_c, q_c)$ they are $p_b q_c$ and $p_c q_b$, $p_c q_a$ and $p_a q_c$, $p_a q_b$ and $p_b q_a$. Thus the points A, B, C of intersection of the three pairs are collinear.

---

**10258.** (Professor NEUBERG.) — On considère toutes les coniques ayant même axe focal $2a$, un foyer fixe F, et le second foyer F′ en un point quelconque d'une droite donnée. Trouver (1) le lieu des sommets de l'axe focal; (2) l'enveloppe des directrices; (3) l'enveloppe de ces coniques.

*Solution by* H. J. WOODALL; Professor SARKAR; *and others.*

1. Since first focus (F) is invariable, and second focus lies on a right line ($f_1$), therefore the centre lies on the right line ($c$) bisecting distance between F and $f_1$; therefore the locus of the vertices consists of the two branches of a *conchoid* of Nicomedes [CARR'S *Synopsis* (5320)].

2. Envelopes of directrices are $y^2 = 4(bx \mp a^2)$, which are parabolæ.

3. Envelope of conics is

$$(x-b)^2 + y^2 = 0 \text{ and } 16a^2\{(x-b)^2 + y^2\} = (4bx - 4a^2 - y^2)^2.$$

[All this follows by the ordinary methods of analytical geometry.]

---

**11328.** (H. FORTEV.)—Find the equation to the evolute of the lemniscata of BERNOULLI. [This problem was also solved by Mr. TUCKER 30 years ago, as Question 1407, in Vol. I. of our series (1st ed., p. 58, 2nd ed., p. 109).]

*Solution by* Prof. MORLEY, M.A.; H. W. CURJEL, B.A.; *and others.*

If $x^2 - 1 = t^2$, where $x$ and $t$ are complex, then, if mod. $t$ is constant, the rectangle of distances of $x$ from $\pm 1$ is constant, and the $x$-path is a Cassinian. In particular, if mod. $t = 1$ the $x$-path passes twice through the $x$-origin, and is a lemniscate. Hence, if we write $y$ = conjugate of $x$,

Digitized by Google

and observe that $1/t$ is the conjugate of $t$, we have the equations of the lemniscate in circular coordinates:

$$x = (1+t^2)^{\frac{1}{2}}, \quad y = (1+t^2)^{\frac{1}{2}}/t.$$

The normal at any point of a curve so given is

$$\begin{vmatrix} x, & y, & 1 \\ x_1, & y_1, & 1 \\ \dot{x}_1, & -\dot{y}_1, & 0 \end{vmatrix} = 0.$$

the dot denotes a differentiation with regard to $t$.

Hence the normal to the lemniscate is $-x + yt^3 + (1+t^2)^{\frac{1}{2}}(1-t^2) = 0$. Differentiating with regard to $t$, and writing $1/t$ for $t$, we have

$$3x = (3+t^2)/(1+t^2)^{\frac{1}{2}}.$$

These are the equations of the evolute in circular coordinates, and from them the curve may be readily discussed.

[Mr. Curjel solves the question thus:—

Equation to the lemniscate is $(x^2+y^2)^2 = a^2(x^2-y^2)$ or $r^2 = a^2\cos 2\theta$,

$$x = a(\cos 2\theta)^{\frac{1}{2}}\cos\theta, \quad y = a(\cos 2\theta)^{\frac{1}{2}}\sin\theta, \quad x^2+y^2 = r^2 = a^2\cos 2\theta.$$

Equation to normal at $(x', y')$ is

$$(x-x')\frac{dx'}{d\theta} + (y-y')\frac{dy'}{d\theta} = 0, \quad \text{or} \quad 2x\frac{dx'}{d\theta} + 2y\frac{dy'}{d\theta} = \frac{dr^2}{d\theta}.$$

That is
$$x\sin 3\theta - y\cos 3\theta - a\sin 2\theta(\cos 2\theta)^{\frac{1}{2}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Differentiating with respect to $\theta$, we get

$$3x\cos 3\theta + 3y\sin 3\theta = \frac{a}{(\cos 2\theta)^{\frac{1}{2}}}(2\cos^2 2\theta - \sin^2 2\theta) \quad \ldots\ldots\ldots (2).$$

Equation to evolute is result of eliminating $\theta$ between (1) and (2). Solving (1) and (2) for $x$ and $y$, we have

$$x = \frac{2a\cos^3\theta}{3(\cos 2\theta)^{\frac{1}{2}}}, \quad y = -\frac{2a\sin^3\theta}{3(\cos 2\theta)^{\frac{1}{2}}}; \quad \text{therefore} \quad \frac{y}{x} = -\frac{\sin^3\theta}{\cos^3\theta};$$

$$x^{\frac{1}{3}} + y = \left(\frac{2a}{3}\right)\left(\frac{1}{\cos 2\theta}\right)^{\frac{1}{3}} = \left(\frac{2a}{3}\right)^{\frac{1}{3}}\left(\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta-\sin^2\theta}\right)^{\frac{1}{3}} = \left(\frac{2a}{3}\right)\left(\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{3}}-y^{\frac{1}{3}}}\right);$$

therefore
$$(x^{\frac{1}{3}}-y^{\frac{1}{3}})(x^{\frac{1}{3}}+y^{\frac{1}{3}}) = \left(\frac{2a}{3}\right)^2.]$$

---

**11184.** (The late W. J. C. Sharp, M.A.)—Prove that

$$\begin{vmatrix} 1, & 1, & 1, & \text{to } n \text{ columns} \\ 1, & 0, & 1, & \ldots \ldots \ldots \\ 1, & 1, & 0, & \ldots \ldots \ldots \\ \ldots & \ldots & \ldots & \ldots \ldots \ldots \end{vmatrix} = (-1)^{n-1}.$$

Digitized by Google

*Solution by* Rev. T. R. Terry, M.A.; Prof. Curtis; *and others.*

It is well known that

$$\begin{vmatrix} x, & a, & a, & \dots & (p \text{ columns}) \\ a, & x, & a, & \dots & \\ a, & a, & x, & & \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} = (x-a)^p \{x+(p-1)a\}.$$

Call this determinant $\phi(x, a, p)$. Then the given determinant

$$= \phi(0, 1, n) + \phi(0, 1, n-1)$$
$$= (-1)^{n-1}\{n-1\} + (-1)^{n-2}\{n-2\} = (-1)^{n-1}.$$

---

**11285.** (A. J. Pressland, M.A.)—Prove that the median triangle of any self-conjugate to a parabola is circumscribed to the parabola.

*Solution by* R. Knowles, B.A.; Professor Cavallin; *and others.*

Let ABC be a triangle of which each vertex is the pole, with respect to the parabola, of its opposite side; $y_1$, $y_2$, $y_3$ the ordinates of ABC; the conditions are complied with if $y_1+y_2 = 2y_3$, $y_2+y_3 = 2y_1$, $y_1+y_3 = 2y_2$, or if $y_1 = y_2 = y_3$; the equation to any side of the median triangle is of the form

$$y-(y_2+y_3)/2 = 2a/y_1\{x-(y_2^2+y_3^2)/8a\} \text{ or } y_1^2-2y_1y+4ax = 0;$$

and its envelope is $y^2 = 4ax$, the original parabola.

---

**11319.** (R. Tucker, M.A.)—D, E, F are points of ex-contact on the sides BC, CA, AB of a triangle. Find the co-ordinates of P, the point of intersection of perpendiculars to the sides at D, E, F. If I be the in-centre, find the equation to the circle on PI as diameter, and derive an expression for $\Sigma\left[a\{(1-\cos A)^2-(\cos B-\cos C)^2\}\right]$.

*Solution by the* Proposer.

The equations to the three perpendiculars at the points of contact are

$$(\cos B-\cos C)\alpha-(1-\cos B)\beta+(1-\cos C)\gamma = 0,$$
$$(1-\cos A)\alpha-(\cos A-\cos C)\beta-(1-\cos C)\gamma = 0,$$
$$(1-\cos A)\alpha-(1-\cos B)\beta-(\cos A-\cos B)\gamma = 0,$$

and the coordinates of P are

$$\alpha_1 = R(1+\cos A-\cos B-\cos C),\ \beta_1 = R(1-\cos A+\cos B-\cos C),$$
$$\gamma_1 = R(1-\cos A-\cos B+\cos C);$$

The circle on PI as diameter is concentric with the circum-circle,

Digitized by Google

therefore its equation is $(a+b+c)(a\beta\gamma+b\gamma a+ca\beta)=(a\alpha+b\beta+c\gamma)^2$;

hence $\Sigma[\ldots]=\begin{bmatrix}a(1+\cos A-\cos B\cos C)\\+b(\ldots)+c(\ldots)\end{bmatrix}^2\Big/(a+b+c)$

$=[a\cos A+b\cos B+c\cos C]^2/(a+b+c)=32S.\sin^2\tfrac{1}{2}A\sin^2\tfrac{1}{2}B\sin^2\tfrac{1}{2}C.$

---

**11080.** (R. Chartres.)—Give an elementary proof of Gregory's Series without the use of $(-1)^{\frac{1}{2}}$.

*Solution by the* Proposer.

Let $\tan\{a_0+a_1x+a_2x^2+\&c.\}=x,$

therefore $\tan\{a_0+a_1(x+h)+a_2(x+h)^2+\&c.\}=x+h;$

therefore $\tan h\{a_1+2a_2x+3a_3x^2+\&c.+\text{terms involving } h, \text{ say } H\}$

$$=\frac{h}{1+x(x+h)};$$

therefore
$$\frac{\tan h\{a_1+2a_2x+3a_3x^2+\&c.+H\}}{h\{a_1+2a_2x+3a_3x^2+\&c.+H\}}$$

$$=\frac{1}{\{1+x(x+h)\}\{a_1+2a_2x+3a_3x^2+\&c.+H\}}.$$

Let $h=0$; then $1=\{(1+x^2)(a_1+2a_2x+3a_3x^2+\&c.)\}^{-1}$; and, equating coefficients of like powers of $x$, therefore $a_1=1$, $a_2=0$, $a_3=-\frac{1}{3}$, &c.; therefore $\tan^{-1}x=a_0+x-\frac{1}{3}x^3+\&c.$, where $a_0=0$, or a multiple of $\pi$.

---

**8842.** (D. Edwardes.)—Prove that

$$\int_0^1\int_0^{\frac{1}{2}\pi}\frac{\log(1-k\sin^2\phi)}{(1-k\sin^2\phi)^{\frac{1}{2}}}\,dk\,d\phi=4\log 2-4.$$

*Solution by* H. J. Woodall; *the* Proposer; *and others.*

Transforming, as in the Solutions of Questions 8465 and 8500, we have

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\sin\theta\cos\phi\log\cos\phi\,d\theta\,d\phi=\iint\frac{\sin\omega\cos\omega}{(1-\sin^2\omega\sin^2\mu)^{\frac{1}{2}}}\log\frac{\cos\omega\,d\mu\,d\omega}{(1-\sin^2\omega\sin^2\mu)^{\frac{1}{2}}}$$

$$=2\log 2-2-\tfrac{1}{4}\int_0^1\int_0^{\frac{1}{2}\pi}\frac{\log(1-k\sin^2\mu)}{(1-k\sin^2\mu)^{\frac{1}{2}}}\,dk\,d\mu;$$

$$\therefore\int_0^1\int_0^{\frac{1}{2}\pi}\frac{\log_e(1-k)}{(1-k\sin^2\phi)^{\frac{1}{2}}}\,dk\,d\phi=2\int_0^1\int_0^{\frac{1}{2}\pi}\frac{\log_e(1-k\sin^2\phi)}{(1-k\sin^2\phi)^{\frac{1}{2}}}\,dk\,d\phi=8\log_e 2-8.$$

Similarly, $\int_0^1\int_0^{\frac{1}{2}\pi}\frac{\{\log_e(1-k)\}^2}{(1-k\sin^2\phi)^{\frac{1}{2}}}\,dk\,d\phi=16+32\left\{\log_e\frac{2}{e}\right\}^2-\frac{4\pi^2}{3}.$

Digitized by Google

**11274.** (G. E. CRAWFORD, M.A.)—If we consider the earth as a perfect sphere irregularly dotted with houses, some one house, say B, will be the farthest of any from a given house A, and some other house, say C, will be the farthest from B. Do A and C necessarily coincide? If not, what is the greatest distance between them? [Distances to be measured along the surface.]

*Solution by* H. W. CURJEL, B.A.; Prof. NILKANTHA SARKAR; *and others.*

Let arc AB = $\alpha$ and let D be diametrically opposite A. The only conclusion we can come to from the fact that B is the house farthest from A, is that there are no houses within the circle, the distance of any point on the circumference of which from D is $\pi - \alpha$ (measured along a great circle). Hence if $\alpha$ is not less than $\frac{1}{2}\pi$, C is the nearest house to the point diametrically opposite to B; therefore the greatest value of AC is $\pi - \alpha$. If $\alpha$ is less than $\frac{1}{2}\pi$, AC is less than $\alpha$. The greatest possible value of AC is when $\alpha = \frac{1}{2}\pi$, and then AC must be not greater than $\frac{1}{2}\pi$.

**11183.** (R. TUCKER, M.A.)—P, Q, R are three cotangential points on the curve $ay^2 = x^3$, and PQ, QR, RP meet the curve again in $r$, $p$, $q$. Prove that $\Delta pqr = 2\Delta PQR$; and find the equation to the circle PQR.

*Solution by the* PROPOSER; Professor GOPHAR; *and others.*

Let the coordinates of a point on the curve be

$$x = a\lambda^2 m^2, \quad y = a\lambda^3 m^3, \quad \lambda \equiv 2/3.$$

Then the equations to the tangent and normal are, respectively,

$$a\lambda^2 m^3 - 3xm + 3y = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$a\lambda^3 m^4 + a\lambda^2 m^2 - my - x = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (1) we see that $\Sigma(m) = 0$. If we call the points P, Q, R, $(m_1, m_2, m_3)$ and denote $\Sigma(m^2)$ by S, and $m_1 m_2 m_3$ by $\mu$, we have the equations to PQ, QR, RP,

$$2m_3 y + \lambda S x = 2a\lambda^3 m_1^2 m_2^2, \ldots, \ldots, \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Hence the point $r$ is $m_1^2 m_2^2/\mu$, $p$ is $m_2^2 m_3^2/\mu$, $q$ is $m_3^2 m_1^2/\mu$. Omitting the factor $a^2\lambda^5/2$, we have

$$\frac{\Delta pqr}{\Delta PQR} = \frac{1}{\mu^5} \begin{vmatrix} 1, & 1, & 1 \\ m_1^4 m_2^4, & m_2^4 m_3^4, & m_3^4 m_1^4 \\ m_1^6 m_2^6, & m_2^6 m_3^6, & m_1^6 m_2^6 \end{vmatrix} \Big/ \begin{vmatrix} 1, & 1, & 1 \\ m_1^2, & m_2^2, & m_3^2 \\ m_1^3, & m_2^3, & m_3^3 \end{vmatrix} = 2 \ldots (4).$$

Expressing the fact that P, Q, R are on the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

we get the equation required to be

$$x^2 + y^2 - \tfrac{1}{4}a\lambda^2 S\,(2 + \lambda^2 S)\,x - 2a\lambda\mu\,(1 + \lambda^2 S)\,y/S + a^2\lambda^4\mu^2\,(2 + \lambda^2 S)/S = 0 \ldots (5).$$

From (3) above the perpendiculars from R on PQ and from P on QR are given by

$$\lambda S y - 2m_3 x = a\lambda^2 m_3^3\,(\lambda^2 S - 2), \quad \lambda S y - 2m_1 x = a\lambda^2 m_1^3\,(\lambda^2 S - 2);$$

Digitized by Google

therefore the orthocentre of PQR is the point

$$-a\lambda^2(\lambda^2 S-2)S/4,\quad a\lambda\mu(\lambda^2 S-2)/2S;$$

the centroid is $a\lambda^2 S/3$, $a\lambda^3\mu$.

---

**11227.** (Professor MANNHEIM.)—On donne un triangle ABC. Du sommet A, on mène la médiane AM qui aboutit au milieu M de BC, et la bissectrice AL qui rencontre la base BC au point L. Du point L, on élève une perpendiculaire à la bissectrice AL; cette droite coupe la médiane au point P et le côté AB au point Q. Démontrer que la perpendiculaire abaissée de P sur BC, et la perpendiculaire élevée de Q à AB, se coupent sur la bissectrice AL.

*Solution by* R. KNOWLES; Professor SARKAR; *and others.*

The equations to AM, AL, LPQ are respectively

$$b\beta-c\gamma=0,\quad \beta-\gamma=0,\quad 2a\alpha+(b-c)\beta-(b-c)\gamma=0,$$

and the coordinates of P, Q are (omitting $2\Delta$)

$$(b-c)^2/a(b+c)^2,\ 2c/(b+c)^2,\ 2b/(b+c)^2,\ -(b-c)/a(b+c),\ 2/(b+c),\ 0$$

the equations to the perpendiculars from P on BC and Q on AB are

$$2a(c\cos B-b\cos C)\alpha-\{2ab+(b-c)^2\cos B\}\beta+\{2ac+(b-c)^2\cos C\}\gamma=0,$$

$$2a\alpha+(b-c)\beta+(c-c\cos A+a\cos B)\gamma=0,$$

and these intersect on the point determined by

$$2a=-(\cos B+\cos C)\beta=-(\cos B+\cos C)\gamma;$$

therefore $\beta=\gamma$, and the point is on AL. [The PROPOSER remarks that —"Si B et C sont les foyers d'une ellipse qui passe par M, les droites menées de P et Q rencontrent la normale AL à cette ellipse au centre de courbure de cette courbe, c'est-à-dire au même point de AL."]

---

**11260.** (Professor CATALAN.)—On donne une droite AB, une circonférence C, et une droite CDP, sur laquelle on prend deux points fixes, D et P. Par le point P, on mène, arbitrairement, une transversale qui coupe en E la droite AB, en F et G la circonférence C. On joint E à D, F et G à C, par les droites DE, CF, CG, qui se coupent en M, M'. Le lieu des points M, M' est une conique, dont on demande la discussion.

*Solution by* Professor GENESE, M.A.

By MENELAUS' theorem, we have $\dfrac{DE.MF}{EM.FC}=\dfrac{DP}{CP}=$ constant.

Digitized by Google

If DH, MN be perpendicular to AB, $\frac{DE}{EM} = \frac{DH}{MN}$; also, DH, FC are constant; therefore MF : MN = constant.

The locus of M is now seen to be a conic, focus at C, directrix a parallel to NH; and the excentricity $= \frac{DP}{CP} \cdot \frac{FC}{DH}$.

---

**9285.** (W. J. Greenstreet, B.A.)—Let P be an angular point of a cube, and let the three faces meeting in P touch the in-sphere in A, B, C; find $r$ of small circle through A, B, C, and the sides and angles of the spherical triangle ABC.

*Solution by* H. J. Woodall.

Obviously sides $= 90°$ each; angles $= 90°$ each; hence we have

$\tan r = \frac{\tan 45°}{\cos (S - A)}$, $S = \frac{3}{4}\pi$, $A = \frac{1}{2}\pi$; therefore $r = \tan^{-1}\sqrt{2}$.

---

**11201.** (G. S. Carr, M.A.)—If $V = f(x, y, p, q, r, \&c.)$, where $y$ is a function of $x$, and $p, q, r, \ldots$ are the successive differential coefficients of $y$; show that the $n^{th}$ variation of V, due to a change in the *form* of the function $y$, is $\delta^n V = \left(\delta y \frac{d}{dy} + \delta p \frac{d}{dp} + \delta q \frac{d}{dq} + \&c.\right)^n V$, where $\delta y$, $\delta p$, &c. follow the law of involution merely, but $\frac{d}{dy}$, &c. the operative law, V being the subject.

*Solution by* Professor G. B. M. Zerr, M.A.

We have
$$\delta V = \delta y \frac{d}{dy} V + \delta p \frac{d}{dp} V + \delta q \frac{d}{dq} V + \&c.$$
$$= \left(\delta y \frac{d}{\delta y} + \delta p \frac{d}{dp} + \delta q \frac{d}{dq} + \&c.\right) V.$$

The factor in parentheses in the right-hand member may be considered constant, and every time we differentiate with respect to $y$, $p$, $q$, &c., we get this same factor; thus, if we differentiate $n$ times, this factor will be raised to the $n$th power; whence the result.

[The reason why the right-hand member in Prof. Zerr's solution may be considered constant is that the variation of $y$ (and therefore also of $p$, $q$, &c.) is assumed to be constant; that is, $\delta y$, $\delta p$, $\delta q$, &c. are constants. Such a function $y$ has been called a *primitive indeterminate function*, that is to say, one whose variation is arbitrary, but of constant form.]

Digitized by Google

**11330.** (Professor Ramaswami Aiyar, M.A.)—If similar figures be described on the faces of an isosceles tetrahedron, prove that (1) any four corresponding points or places determine an isosceles tetrahedron; and (2) any four corresponding lines are generators of the same system of a hyperboloid.

---

*Solution by* Professors Schoute, Bhattacharya, *and others.*

Four vertices P, Q, R, S (Fig. 1) of a rectangular parallelopipedon, no two of which are joined by an edge of this figure, form the vertices of an isosceles tetrahedron.

It is evident that a rotation of $\pi$, about one of the three rectangular axes $OA_1$, $OA_2$, $OA_3$, interchanges the *congruent* figures $F_1$, $F_2$, $F_3$, $F_4$

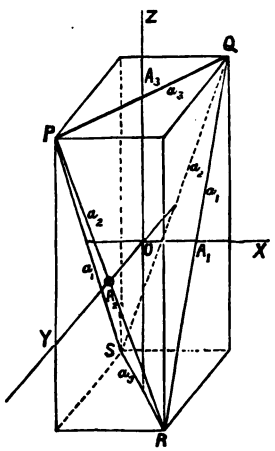



Fig. 1.

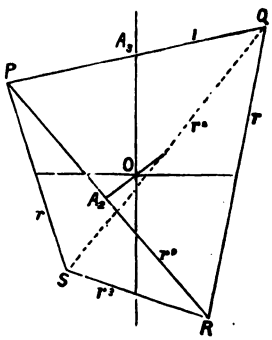



Fig. 2.

determined by the congruent faces SRQ, RSP, QPS, PQR of the tetrahedron.

So the semi-revolution about

$$\left.\begin{matrix} OA_1 \\ OA_2 \\ OA_3 \end{matrix}\right\} \text{ interchanges } \left\{\begin{matrix} F_1 \text{ and } F_2, & F_3 \text{ and } F_4 \\ F_1 \text{ ,, } F_3, & F_4 \text{ ,, } F_2 \\ F_1 \text{ ,, } F_4, & F_2 \text{ ,, } F_3 \end{matrix}\right\}.$$

The mid-point O is a common centre of the four figures.

With respect to the axes $OA_1$, $OA_2$, $OA_3$, any four corresponding points can be represented by the coordinates

$$-x, y, z; \quad x, -y, z; \quad x, y, -z; \quad -x, -y, -z.$$

This proves the first part of the theorem. For the coordinates of P, Q, R, S are expressed in quite the same manner with determinate values of $x$, $y$, $z$. The volume of the new tetrahedron is $\frac{4}{3}xyz$. If one of the three coordinates $x$, $y$, $z$ vanishes, the four points are complanar.



Digitized by Google

By the three semi-revolutions, the line $y = ax + b,\ z = cx + d$ is brought into the positions

$$\left.\begin{aligned} -y &= ax + b \\ -z &= cx + d \end{aligned}\right\}, \qquad \left.\begin{aligned} y &= -ax + b \\ -z &= -cx + d \end{aligned}\right\}, \qquad \left.\begin{aligned} -y &= -ax + b \\ z &= -cx + d \end{aligned}\right\}.$$

These four lines belong to the hyperboloid

$$(ad - bc)(acx^2 - bd) = cdy^2 - abz^2;$$

this proves the second part of the theorem. On the hyperboloid, the four generators form the anharmonic rapport $bc/ad$ (or one of the other five corresponding values). For the lines of the tetrahedral complex $bc + ad = 0$, the quadruples are harmonical, &c.

[In a memoir, "On Tetrahedrons with Similar Faces" (*Verslagen en mededeelingen der Koninklijke Akademie von Wetenschappen, Afdeeling Natuurkunde*, 3de reeks, deel. VI., bladzij 462), it is shown that there is a second kind of isosceles tetrahedrons, the four trihedrons of which are isosceles. The three pairs of opposite edges of such a tetrahedron (Fig. 2) can be represented by $(1, r^3)$, $(r, r)$, $(r^2, r^2)$. For the *similar* figures, $F_1$, $F_2$, $F_3$, $F_4$, determined by the faces SRQ, &c., that form two similar pairs of congruent figures, the two theorems do not hold.]

---

**11388.** (The Bachelors of Cambridge)—

In ninety-two and ninety-three
Twelve balls of ours we fix to be:
Ten Girton maids, Urania's pets,
We bid, in unnamed quintuplets,
To each to come,
Five left at home,
While dance a dozen different sets.
The beauties don't deny, but all
Cry: "Most unmathematical!
Shall we, for random invitations,
Descend to work at Combinations?
Nay—let the males, to learn themselves,
Pile divers fives of us, in twelves
Of fives, till found by skill or chance is
Each separate twelve, that gives the dances
To every name
In tale the same."
The saucy Ten's demand,
That they may fit their fancies
With all the dozens of fives in hand—
We are bound to find an answer to it;
But, O ye Muses, how to do it?

*Solution by* H. J. Woodall.

(1) ABCDE, (2) FGHJK, (3) AGCJE, (4) FBHDK, (5) AHEGD, (6) FCKBJ, (7) AJBKC, (8) FDGEH, (9) AKBFC, (10) GDHEJ, (11) AFBGC, (12) HDJEK.

This, obtained by skill, is a solution. But difficulties arise because of A's nine companions. A selection must be made to take the 24 invitations which are issued with A's 6 invitations, and 24 cannot be evenly distributed among 9. If there had been 18 parties, A would be at 9, and each of her companions would have had 4 invitations with A.

---

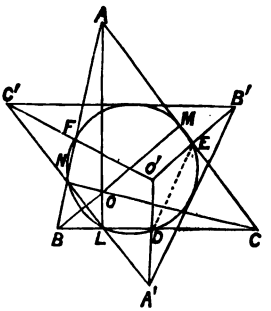
**11269.** (Professor Sarkar.)—O is the orthocentre of a triangle ABC, and O′ is the centre of its circumscribing circle; A′, B′, C′ are the centres of the circles circumscribing the triangles BOC, COA, AOB respectively.

Digitized by Google

Show that O′ is the orthocentre of A′B′C′, O the centre of its circumscribing circle, A, B, C those of the circles circumscribing B′O′C′, C′O′A′, A′O′B′ respectively, and that all the eight triangles above mentioned have the same nine-point circle.

---

*Solution by* C. MORGAN, M.A.; Prof. RATNAM AIYAR, B.A.; *and others.*




BOC = 180° − A. Therefore radius of circle round BOC = radius of circle round ABC. (Let D, E, F be middle points of sides; therefore O′D, A′D are each perpendicular to BC.) Then we have

$$CA' = CO' = CB';$$

∴ C is centre of circle round A′O′B′.

Also, $A'O = CO' = B'O$;

∴ O′ is centre of circle round A′B′C′.

The triangles CO′D, CA′D are clearly equal in all respects; therefore O′D = A′D. So O′E = A′E; therefore DE is parallel to A′B′; therefore A′B′ is parallel to AB. So B′C′ is parallel to BC; hence A′O′, which is perpendicular to BC, is also perpendicular to B′C′; hence O′ will will be the orthocentre of A′B′C′.

The nine-point circle of ABC passes through L, D, M, and is therefore the nine-point circle of BOC.

It also passes through D, E, F, which have been shown to be middle points of O′A′, O′B′, O′C′ (where O′ is the orthocentre of A′B′C′), and is therefore the nine-point circle of A′B′C′, and therefore goes through middle point of A′B′.

It also passes through D, E and middle point of A′B′, and is therefore nine-point circle of A′O′B′.

---

**8675.** (By Professor HAUGHTON, F.R.S.)—The daily energy of the average diet of all the armies of Europe is estimated at 3694 foot-tons; the daily work done for this expenditure of food (including the work done in moving the man's own body) is estimated at 430·6 foot-tons. If man be regarded as a perfect heat-engine, whose upper temperature is 100° F. (blood-heat), calculate from the foregoing data the lower temperature at which the engine is worked.

---

*Solution by* H. J. WOODALL.

Efficiency $= 430{\cdot}6/3694 = (t - t')/t$, $t$ = higher temperature, $t'$ = lower temperature on absolute scale;

*e.g.*, if $t$, $t'$ are degrees C., then $t = T + 273$, $t' = T' + 273$, and $T = 37{\cdot}8°$ C. (= 100° F.);

∴ $T' = T - t \times 430{\cdot}6/3694 = 37{\cdot}8° - 36{\cdot}2° = 1{\cdot}6°$ C. = 35° F., about.

Digitized by Google

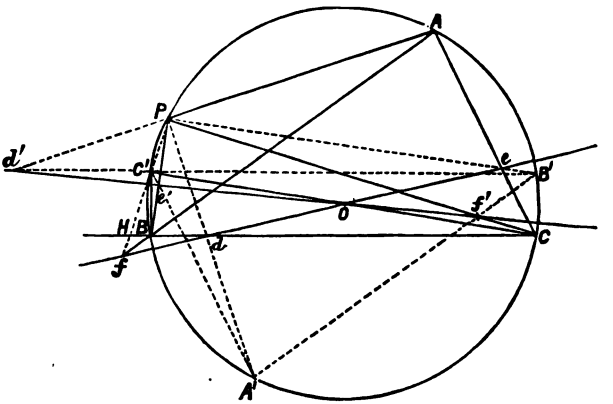
**11343.** (R. Tucker, M.A.)—ABC is a circle. ABC, A′B′C′ are two diametral triangles in the circle. Any point (P) on the circumference is joined to the six points; so that PA′, PB′, PC′ cut BC, CA, AB in $d, e, f$; and PA, PB, PC cut B′C′, C′A′, A′B′ in $d', e', f'$. Prove that $def$, $d'e'f'$ coincide with two diameters of the circle.

---

*Solution by the* Proposer.

Let P′C meet CB in H, and assume $\angle PAB = \phi$; then, from $\triangle CHC'$, we have $C'O/CO = 1$, $Cd/dH = \cot\phi \cot B$, $Hf/C'f = \tan\phi \tan B$ ... (i.);

$$\therefore \quad C'O \,.\, Cd \,.\, Hf = CO \,.\, dH \,.\, C'f;$$




therefore $Odf$ is a straight line. Hence $fdOe$ is a straight line. Similarly, $f'd'Oe'$ is a straight line.

(i.) is got thus: $Cd/dP = \cos B/\sin\phi$, $dP/dH = \cos\phi/\sin B$,

and $\quad Hf/Bf = \sin B/\cos\phi$, $\quad Bf/C'f = \sin\phi/\cos B$;

or thus: in triangles $BfH$, $dPH$, we have $\angle HBf = B = \angle dPH$, and angles $dHf$, $dHP$ supplementary;

$$\therefore \quad Hf \,.\, dP = Bf \,.\, dH; \qquad \text{and} \quad Cd \,.\, Bf = dP \,.\, C'f.$$

We can prove $def$ collinear thus:

$$\angle BPd = \angle BAA' = 90° - C, \quad \angle CPd = \angle CAA' = 90° - B,$$

$$\frac{Bd}{Cd} = \frac{BP\cos C}{CP\cos B}, \quad \frac{Ce}{Ae} = \frac{CP\cos A}{AP\cos C}, \quad \frac{Af}{Bf} = \frac{AP\cos B}{BP\cos A};$$

$$\therefore \quad Af \,.\, Bd \,.\, Ce = Ae \,.\, Bf \,.\, Cd; \qquad \therefore \quad def \text{ is a straight line.}$$

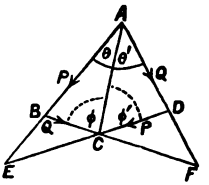
---

**11336.** (Professor Genese, M.A.)—The bisectors of the angles between opposite sides produced of a quadrilateral in a circle meet on the join of the mid-points of the diagonals.

Digitized by Google

*Solution by* W. J. Johnston, M.A.; James Rice; *and others.*

Let equal forces P, P act along AB, DC; and equal forces Q, Q along AD, BC. Then the resultant of the system passes through the intersection of the bisectors of the angles E, F.




Again, its moment about mid-point of AC,

$$\begin{aligned} M &= P.\tfrac{1}{2}AC\sin\theta + Q.\tfrac{1}{2}BC\sin\phi \\ &\quad - P.\tfrac{1}{2}CD\sin\phi' - Q.\tfrac{1}{2}AC\sin\theta' \\ &= P.\tfrac{1}{2}AC.BC/2R + \&c. \\ &= (P.AC)/4R.(BC-AD) - (Q.AC)/4R.(CD-AB). \end{aligned}$$

Let $P = k(CD-AB)$, $Q = k(BC-AD)$. Then $M = 0$, and, by symmetry, the moment of the system about the mid-point of BD also annuls itself; therefore, &c.

---

**4979.** (Professor Hudson, M.A.)—If a conical cup be filled with liquid, prove that the mean pressure at a point in the volume of the liquid is to the mean pressure at a point in the surface of the cup as 4 : 3.

*Solution by* Professor Schoute.

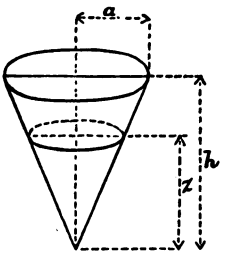
If $\rho$ is the density of the fluid, the mean pressure in the volume is

$$\rho\int_0^h (h-z)\,z^2dz \Big/ \int_0^h z^2dz = \tfrac{1}{4}\rho h.$$

The mean pressure at the surface is

$$\rho\int_0^h (h-z)\,zdz \Big/ \int_0^h zdz = \tfrac{1}{3}\rho h;$$

$$\therefore\ p_v : p_s = 3 : 4.$$




---

**8274.** (Professor Haughton, F.R.S.)—Owen Harte has performed the astonishing feat (1885) of throwing the 16 lb. hammer the distance of 117 ft. 3½ in. Calculate in foot-pounds the work done by his muscles in this effort.

*Solution by* H. J. Woodall.

We have $117\frac{7}{4} = d = v^2\sin 2a/g$; $\therefore\ v^2/g = 117\frac{7}{4}/\sin 2a$.

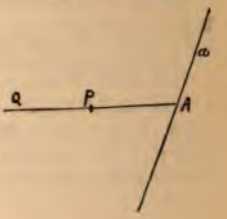
Work done $= \frac{1}{2}Mv^2 = \frac{1}{2}Wv^2/g = \frac{16}{2} \times 117\frac{7}{4}/\sin 2a = 938\frac{1}{3}/\sin 2a$,

where $a$ = angle with the horizon at which the hammer left his hand.

Digitized by Google

**11387.** (The Editor.)—If a parabola move parallel to itself in such wise that its vertex is always on a fixed line, prove that the locus of the pole of a fixed straight line with respect to the parabola is a straight line.

*Solution by* H. W. Curjel, B.A.; Prof. Radhakinshuan; *and others.*

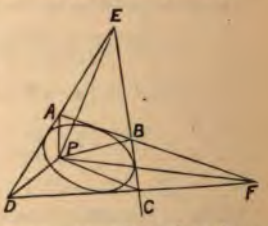



Since the parabola always moves parallel to itself, the diameter conjugate to the fixed straight line is fixed in the parabola, as also is the vertex, P say, of this diameter. P describes a straight line parallel to that which is described by the vertex of the parabola.

Let $a$ be the fixed straight line. Draw PA parallel to the axis of the parabola, meeting $a$ in A, and produce AP to Q, making PQ = AP. Then Q is the pole of $a$. But the locus of P is a straight line. Hence it is evident that the locus of Q is also a straight line.

**11391.** (R. Lachlan, M.A.)—A conic is inscribed in a quadrilateral, and three other conics are drawn to osculate the given conic at the same point and pass respectively through the extremities of the diagonals of the quadrilateral. Show that the poles with respect to these conics of the three diagonals are coincident; and that the three conics and the given conic have another common tangent.

*Solution by* A. E. Jolliffe; Professor Bhattacharya; *and others.*




Let P be the point of osculation. The harmonic conjugate of the tangent at P, with respect to the pairs PA, PC; PB, PD; PE, PF, is the same, and on this straight line lie the poles in the question. But, because of the osculation at P, they lie also in the consecutive positions of the straight line. Therefore all three coincide in the point where this touches its envelope.

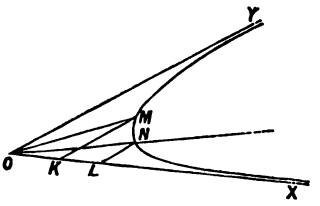
Let O be this point. Then the pairs of tangents from O to the four conics are in involution, and they have three tangents common; therefore they have a fourth.

**11411.** (Professor Neuberg.)—On donne une suite d'hyperboles ayant les mêmes asymptotes OX, OY. On prend sur chacune de ces courbes un point M tel que le secteur compris entre la courbe, la droite OM, et

Digitized by Google

une droite fixe menée par O, ait une aire constante. Trouver le lieu du point M.

*Solution by* H. W. Curjel, B.A.; F. W. A. Clarke; *and others.*




Let the given area = A, and let $\omega$ = angle between asymptotes.

Let one of the hyperbolæ $xy = c^2$ cut the given straight line $y = mx$ in N $(x_0, y_0)$; hence, if $x, y$ be the coordinates of M, the sectorial area

$$\text{NOM} = c^2 \sin\omega \int \frac{dx}{x} = c^2 \sin\omega \log \frac{x}{x_0};$$

$$\therefore \text{A} = c^2 \sin\omega \,.\, \log xm^{\frac{1}{2}}/c$$

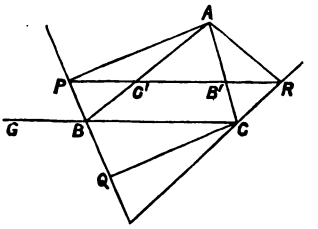
$$= xy \sin\omega \,.\, \log xm^{\frac{1}{2}}/(xy)^{\frac{1}{2}};$$

thus the required locus is $xy \sin\omega \,.\, \log xm/y = 2\text{A}$.

---

**11384.** (Professor Nixon, M.A.)—If A is the intersection of a direct and a transverse common tangent to two circles, and AL perpendicular to their line of centres, show that L is a limiting point of the circles. Deduce *geometrically* that, if D is the diameter of the circle cutting the three ex-circles of a triangle orthogonally, $r$ the in-radius and $s$ the semi-perimeter of the triangle, then $\text{D}^2 = r^2 + s^2$. [A longer proof than here indicated of this theorem, which forms Question 8343, is given by Mr. Davis in Vol. xlviii. of our *Mathematics*.]

*Solution by* W. J. Johnston, M.A.; R. Henderson, B.A.; *and others.*




As L is the intersection of two diagonals of the quadrilateral circumscribing the circles, it has the *same* polar with regard to both circles, viz., the third diagonal; therefore L is a limiting point.

Let ABC be the triangle, A′B′C′ the mid-points of its sides.

Let AP, CQ be perpendiculars on the external bisector of $\angle$ B; AR on that of $\angle$ C.

By the preceding, the orthogonal circle passes through P, Q, R. Let $i$ be its centre. Then $i$ is on the perpendicular to PQ through its mid-point. This passes through B′, and is parallel to internal bisector of $\angle$ B; therefore $i$ is the incentre of A′B′C′.

Again, joining PC′, $\text{PC}' = \frac{1}{2}\text{AB} = \frac{1}{2}c$; also, $\angle$ C′PB = C′BP = GBP; therefore PC′ is parallel to BC; PC′, C′B′, B′R are collinear, and PR = $s$;

$$\therefore (\tfrac{1}{2}\text{D})^2 = (\tfrac{1}{2}\text{PR})^2 + (\text{perp. from } i \text{ on PR})^2 = (\tfrac{1}{2}s)^2 + (\tfrac{1}{2}r)^2,$$

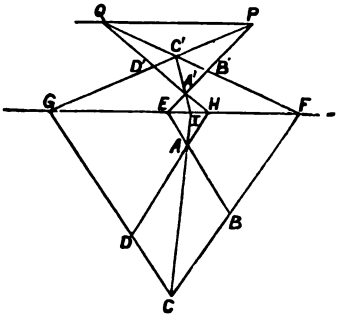
or

$$\text{D}^2 = r^2 + s^2.$$

Digitized by Google

**11171.** (Professor NIEWENGLOWSKI.)—Par un point P donné dans le plan d'un parallélogramme ABCD, mener, avec la règle seule, une parallèle à une droite $mn$ située dans ce plan.

*Solution by* J. C. ST. CLAIR; Professor SARKAR; *and others.*




Produce the sides and one of the diagonals of the given parallelogram to meet the given line EF in E, F, G, H, I respectively. Through I draw any line cutting EP in A' and GP in C'. Let HA', FC' meet in Q; then PQ is parallel to EF.

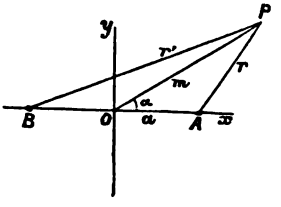
For if B' denote the intersection of EP, FQ, and D' that of GP, HQ, the figures ABCD, A'B'C'D' are homological, EF being the axis of homology. The line PQ in the second figure corresponds to the line at infinity in the first figure; accordingly it is the vanishing line of the second figure, and is consequently parallel to EF.

[This problem is solved in PONCELET'S *Propriétés Projectives*, Tome I., 198; also by LAMBERT, in CREMONA'S *Projective Geometry*, 115.]

---

**4969.** (R. A. ROBERTS, M.A.)—If, from any point of a Cassinian oval, tangents be drawn to the curve, prove that their six points of contact lie on a conic.

*Solution by* Professor SCHOUTE.




The theorem is proved when we show that the focus F of the circular cubic that forms the first polar curve of the point P on the bicircular Cassinian oval, with respect to that oval, is situated on the normal in P at that quadric, and therefore at the cubic too. For in this case the circle with the centre F and the radius FP, contains six of the twelve points common to the Cassinian oval and the cubic. So, according to a well-known theorem, the other six lie on a conic.

We determine the oval by the poles A, B and the point P. Then the well-known relation, $rr' =$ constant, is transformed in

$$(x^2+y^2)^2-2a^2(x^2-y^2)-m^2(2a^2\cos 2\alpha-m^2)=0,$$

Digitized by Google

and the first polar of P becomes

$$m(x^2+y^2)(x\cos\alpha+y\sin\alpha)-a^2(x^2-y^2)\ldots=0.$$

In the common way, we find, for the two imaginary asymptotes and the coordinates of their common point,

$y=ix-(a^2i/m)e^{-i\alpha}$, $y=-ix+(a^2i/m)e^{i\alpha}$, $x=a^2/m\cos\alpha$, $y=-a^2/m\sin\alpha$.

Now the tangent to the oval in P is

$$(m^2-a^2)\cos\alpha\,.\,x+(m^2+a^2)\sin\alpha\,.\,y+\ldots=0;$$

hence the directional coefficients of FP and of this tangent are

$$\frac{m\sin\alpha+a^2/m\sin\alpha}{m\cos\alpha-a^2/m\cos\alpha}\quad\text{and}\quad-\frac{(m^2-a^2)\cos\alpha}{(m^2+a^2)\sin\alpha},\quad\&c.$$

[The theorem above referred to is this:—When $np-\frac{1}{2}(p-1)(p-2)$ of the points common to $C_n$ and $C'_n$ lie on a curve $C_p$ $(p<n)$, this curve $C_p$ will contain $np$ of these points, and the other $n(n-p)$ points common to $C_n$ and $C_n'$, will lie on a curve $C_{n-p}$. According to this addition, the two new points common to the oval and the conic of the problem and the two new points common to the oval and the circle of the solution are collinear.]

---

**10064.** (Professor Bordage.)—Examine the surface that has for equation $e^{-z}=\cos x\cos y$, and find its lines of curvature.

---

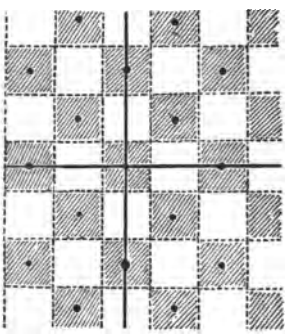
*Solution by* H. J. Woodall.

Surface is $e^{-z}=\cos x\cos y$.

$z=0$ is a limiting plane, and touches where $\cos x\,.\cos y=1$, *i.e.*, at points $\begin{pmatrix}x=\pm2n\pi\\y=\pm2m\pi\end{pmatrix}$ and $\begin{pmatrix}x=\pm(2n+1)\pi\\y=\pm(2m+1)\pi\end{pmatrix}$;

$z=\infty$ cuts where $\cos x\,.\cos y=0$, *i.e.*, along lines $x=\pm\frac{1}{2}(2n+1).\pi$,

$y=\pm\frac{1}{2}(2m+1).\pi$.



At intersection of plane,

$x=2n\pi$, $\cos x=1$; $\therefore\ e^{-z}=\cos y$;

$\therefore-\frac{dz}{dy}e^{-z}=-\sin y$; $\therefore\frac{dz}{dy}=\tan y=0$;

therefore $z=0$ is tangential to surface.

The accompanying figure represents the plan of the surface: the dots are the apices, the shaded squares show the square prisms of space (of length to infinity) which contain the surface, the dotted lines are the lines in which the plane infinity is cut.

$$U\equiv f(x,y,z)\equiv e^{-z}-\cos x\cos y=0,\quad dU/dx=\sin x\cos y,$$

$$dU/dy=\cos x\sin y,\quad dU/dz=-e^{-z}=-\cos x\cos y,$$

$A=\tan x$, $B=\tan y$, $C=-1$, $dA=\sec^2x\,dx$, $dB=\sec^2y\,dy$, $dC=0$.



Digitized by Google

$$\therefore \quad \begin{vmatrix} dx, & dy, & dz \\ \tan x, & \tan y, & -1 \\ \sec^2 x\,dx, & \sec^2 y\,dy, & 0 \end{vmatrix} = 0;$$

$$\therefore \quad dx \begin{vmatrix} \tan y, & -1 \\ \sec^2 y\,dy, & 0 \end{vmatrix} - dy \begin{vmatrix} \tan x, & -1 \\ \sec^2 x\,dx, & 0 \end{vmatrix} + dz \begin{vmatrix} \tan x, & \tan y \\ \sec^2 x\,dx, & \sec^2 y\,dy \end{vmatrix} = 0$$

$$\therefore \sec^2 y\,dx\,dy - \sec^2 x\,dx\,dy - dz\,(\tan y \sec^2 x\,dx - \tan x \sec^2 y\,dy) = 0,$$

$$dx\,dy\,(\sec^2 y - \sec^2 x) + dz\,(\tan x \sec^2 y\,dy - \tan y \sec^2 x\,dx) = 0.$$

$$\frac{dz}{dx} e^{-z} = +\sin x \cos y + \cos x \sin y \frac{dy}{dx}; \quad \therefore \quad \frac{dz}{dx} = \tan x + \tan y \frac{dy}{dx};$$

$$\therefore \frac{dy}{dx}(\sec^2 y - \sec^2 x) + \left(\tan x + \tan y \frac{dy}{dx}\right)\left(\tan x \sec^2 y \frac{dy}{dx} - \tan y \sec^2 x\right) = 0;$$

$$\therefore \tan x \tan y \sec^2 y \left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx}(\sec^2 y - \sec^2 x + \tan^2 x \sec^2 y - \tan^2 y \sec^2 x)$$
$$- \tan x \tan y \sec^2 x = 0;$$

whence $\sec y \dfrac{dy}{dx} \pm \sec x = 0; \quad \therefore \quad \dfrac{dy}{\cos y} \pm \dfrac{dx}{\cos x} = 0.$

Solving, $\log \tan (\frac{1}{4}\pi - \frac{1}{2}y) \pm \log \tan (\frac{1}{4}\pi - \frac{1}{2}x) = \log A;$

$$\therefore \tan (\tfrac{1}{4}\pi - \tfrac{1}{2}y) \tan (\tfrac{1}{4}\pi \pm \tfrac{1}{2}x) = A.$$

The intersection with the surface $e^{-z} = \cos x \cos y$ gives the lines of curvature required.

---

**11362.** (W. W. Johnson.)—Given three points and three straight lines in a plane, the determinant of the nine perpendiculars from the points to the lines is equal to twice the product of the areas of the triangles formed by the points and by the lines, divided by the radius of the circle circumscribing the latter.

*Solution by* Professor Schoute.

If the three lines, in Hesse's form, are

$x \cos \alpha + y \sin \alpha - p = 0, \quad x \cos \beta + y \sin \beta - q = 0, \quad x \cos \gamma + y \sin \gamma - r = 0,$

and the three points $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$; the determinant of the nine perpendiculars is the product of the two determinants

$$\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix}, \qquad \begin{vmatrix} \cos \alpha, & \sin \alpha, & p \\ \cos \beta, & \sin \beta, & q \\ \cos \gamma, & \sin \gamma, & r \end{vmatrix}.$$

The former of these two represents twice the area of the triangle formed by the points. The latter equals

$p \sin (\gamma - \beta) + q \sin (\alpha - \gamma) + r \sin (\beta - \alpha)$ or $p \sin A + q \sin B + r \sin C,$

where ABC is the triangle formed by the lines. If O is the circumcentre of ABC and $AO = \rho$, it is easily seen that $p \,.\, \rho \sin A$ represents the area of the triangle BCO. Therefore

$$p \sin A + q \sin B + r \sin C = \frac{\text{area ABC}}{\rho}, \text{ \&c.}$$

Digitized by Google

**10267.** (Professor HERMITE.) — On sait que les coefficients de la puissance $m$ du binôme sont divisibles par $m$, à l'exception du premier et du dernier, lorsque $m$ est un nombre premier. (1) On propose d'étendre cette proposition en montrant que l'expression (A)

$$\{m(m-1)\dots(m-n+1)\}/n!$$

est divisible par $m/\delta$, $\delta$ désignant le plus grand commun diviseur de $m$ et $n$. (2) Soit $\epsilon$ le plus grand commun diviseur des nombres $m+1$ et $n$; on demande d'établir que le nombre entier (A) est divisible par $(m-n+1)/\epsilon$.

---

*Solution by* H. J. WOODALL.

(B) $= \{(m-1)(m-2)\dots(m-n+1)\}/(n-1)!$ is whole (known theorem). Multiply by $m/n = m/\delta n_1$; it is still whole. Therefore $n_1$ divides the expression (B) and $m/\delta$ divides (A).

Again (C) $= \{m(m-1)(m-2)\dots(m-n+2)\}/(n-1)!$ is whole. Multiply by $(m-n+1)/n = (m-n+1)/\epsilon n_2$; it is still whole. Therefore $n_2$ divides (C) and $(m-n+1)/\epsilon$ divides A.

---

**10231.** (Professor DE WACHTER.)—If $a$, $b$, $c$ be the radii of gyration of a given plane area about three axes perpendicular to its plane in A, B, C, show that, if circles be drawn with centres A, B, C, and radii $a$, $b$, $c$, respectively, their radical centre will be the area's centroid.

---

*Solution by* H. J. WOODALL.

Let $k$ be radius of gyration about the centroid O; $d_1$, $d_2$, $d_3$ be distances of A, B, C from O respectively. Then it is known {CARR (5921)} that

$$a^2-d_1^2 = b^2-d_2^2 = c^2-d_3^2 = k^2.$$

But this shows that the tangents from O to the circles are of equal lengths $= k$, whence the theorem.

[This theorem gives a method for finding the centroid, and thence the radius of gyration about that point.]

---

**11189.** (Rev. T. R. TERRY, M.A.)—If

$$N = (x-a)^5(b-c)^5 + (x-b)^5(c-a)^5 + (x-c)^5(a-b)^5,$$

and

$$D = (x-a)^3(b-c)^3 + (x-b)^3(c-a)^3 + (x-c)^3(a-b)^3,$$

where $a$, $b$, $c$ are the roots of the equation $y^3+3Py^2+3Qy+R=0$; show that

$$\frac{N}{15D} = (P^2-Q)x^2 + (PQ-R)x + (Q^2-PR).$$

Digitized by Google

*Solution by the* PROPOSER.

Both N and D vanish when $b = c$ or when $x = a$; therefore

$(a-b)(b-c)(c-a)(x-a)(x-b)(x-c)$ is a factor of each;

hence N is divisible by D, the quotient being of the form $Ax^2+Bx+C$, where A, B, C are respectively of 2, 3, 4 dimensions in $a, b, c$. Now A, B, C may be found as follows:—

Let $$b-c=-p, \quad x-c=X, \quad c-a=-q,$$
$$x-a=X-q, \quad a-b=p+q, \quad x-b=X+p.$$

Then

$$D=(p+q)^3X^3-p^3(X-q)^3-q^3(X+p)^3=3pq(p+q)X(X+p)(X-q);$$

$$N=(p+q)^5X^5-p^5(X-q)^5-q^5(X+p)^5$$
$$=5pq(p+q)X(X+p)(X-q)\left[(p^2+pq+q^2)X^2-pq(p-q)X+p^2q^2\right].$$

Therefore $$3N/(5D)=(a^2+b^2+c^2-ab-bc-ca)x^2$$
$$-(b^2c+bc^2+c^2a+ca^2+a^2b+ab^2-6abc)x+b^2c^2+c^2a^2+a^2b^2-abc(a+b+c);$$

whence the required result.

---

**3964.** (The late M. COLLINS, B.A.)—Show that the Chaldean Saros, or period of 223 lunations (= 6585·3212 days), cannot be safely employed oftener than 49 or 50 times successively in predicting lunar eclipses.

*Solution by* H. J. WOODALL.

Suppose that at a certain epoch when the moon crosses the ecliptic the sun is at A. When the moon again crosses the ecliptic (at the end of the Saros) the sun will be at B, and distance AB = 28′. But if there is to be an eclipse, the distance between the moon and the point opposite the sun must be less than 12°. Accordingly the same eclipse is repeated $2 \times 12 \times 60/28$ = about 50 times.

---

**2442.** (The late Professor TOWNSEND, F.R.S.)—Through a given point in a plane draw a line the sum of the squares of whose distances from any number of given points in the plane shall be a maximum, a minimum, or given.

*Solution by* H. J. WOODALL.

Let $(x-h)\cos\theta+(y-k)\sin\theta=0$ be the line through $(h, k)$, if $(x, y)$, &c., are the points.

$$U=\Sigma p^2=A\cos^2\theta+H\sin 2\theta+B\sin^2\theta \quad \ldots\ldots(1),$$
$$\frac{dU}{d\theta}=2\sin 2\theta(B-A)+2\cos 2\theta . H \quad \ldots\ldots(2).$$

Therefore, for maximum or minimum, (2) = 0; $\therefore \tan 2\theta = H/(A-B)$. If U is given, we must solve (1).

Digitized by Google

**11229.** (Professor NEUBERG.)—Par un point quelconque P situé à l'intérieur du triangle donné ABC, on mène une parallèle à BC, coupant CA en C′ et AB en B″; une parallèle à CA, coupant AB en A′ et BC en C″; une parallèle à AB, coupant BC en B′ et CA en A″. La probabilité que les segments B′C″, C′A″, A′B″ soient les côtés d'un triangle, est

$$p = 2abc/\{(b+c)(c+a)(a+b)\}.$$

*Solution by* Professors ZERR, MUKHOPADHYAY, *and others.*

Let D, E, F be the points where the bisectors of the angles meet the opposite side. Then, if P be anywhere within the triangle DEF, the sum of any two of the segments in question will be greater than the third, and the segments will be the sides of a triangle. Hence we have

$$p = \text{area DEF} : (\text{ABC});$$

$$\text{area DEF} = \text{ABC}\left\{\frac{2abc}{(b+c)(c+a)(a+b)}\right\};$$

therefore $p = 2abc/\{(b+c)(c+a)(a+b)\}$;

and if $a = b = c$, $p = \frac{1}{4}$.

---

**8145.** (Rev. T. R. TERRY, M.A.)—If $x^2+y = 18$, and $x+y^2 = 8$, show that all the values of $x$ are real; find roughly their position; and calculate the greatest value of $x$ to six places of decimals.

*Solution by* H. J. WOODALL.

We have $x^4-36x^2+x+316 = 0$, or $(x-4)(x^3+4x^2-20x+79) = 0$,

$$u \equiv x^3+4x^2-20x+79 = 0;$$

| $x$ | $w$ | $x$ | $u$ |
|---|---|---|---|
| 0 | − | −1 | − |
| 1 | − | −2 | − |
| 2 | − | −3 | −10 |
| 3 | − | | |
| 4 | − | −4 | +1 |
| 5 | + | −5 | − |
| $+\infty$ | + | $-\infty$ | − |

roots of $u$ lie when $x$ is between

+4 and +5,
−3 and −4,
−4 and −5;

the former is 4·458899,

the latter is −4·182.

---

**11340.** (Professor DE LONGCHAMPS.)—On considère une parabole P dont le sommet est S. Soit AA′ une corde principale de P. Il existe une circonférence Δ tangente, aux points A, A′, aux droites SA, SA′; elle coupe P en deux points B, B′, différents des points A, A′. Démontrer

Digitized by Google

que (1) la distance des droites AA′, BB′ est égale au paramètre $2p$; (2) ayant tracé une droite parallèle à l'axe et touchant $\Delta$ en I, le lieu de ce point, quand AA′ est mobile, est la parabole P; et (3) trouver l'enveloppe des circonférences telles que $\Delta$.

---

*Solution by* R. Knowles, B.A.; Professor Aiyar, M.A.; *and others.*

Let $x-x_1=0$ be the equation to AA′; those to SA, SA′ and the perpendiculars to them at A, A′ are respectively $x_1y-y_1x=0$; $x_1y+y_1x=0$; $x_1x+y_1y=x_1(x_1+2p)$; $x_1x-y_1y=x_1(x_1+2p)$; therefore the coordinates of the centre of the circle are $x_1+2p$, $0$; and its equation is

$$x^2+y^2-2(x_1+2p)x+x_1(x_1+2p)=0.$$

Put $x=y^2/2p$ in this, and in $x-x_1=0$, and divide; then

$$\{y^4-4p(x_1+p)y^2+4p^2x_1(x_1+2p)\}/(y^2-2px_1)=y^2-2px_1-4p^2.$$

Put $y^2=2px$; then $x=x_1+2p$ is the equation to BB′, and the distance is $=2p$; (2) putting $y^2=2p(x_1+2p)$ in the equation to the circle, we get $\{x-(x_1+2p)\}^2=0$; therefore the diameter $y=[2p(x_1+2p)]^{\frac{1}{2}}$ touches the circle and meets the parabola in the same point I; (3) the envelope of the circle is the parabola $y^2-4px-(1-p)^2=0$.

---

**11349.** (A. S. Ramsay, B.A.)—If $\rho$, $\rho'$, and $2n$ are respectively the principal radii of curvature and the length of the normal chord at a point on the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, the length of the central perpendicular on the tangent plane at which is $p$; prove that (1)

$$\frac{1}{\rho}+\frac{1}{\rho'}+\frac{1}{n}=p\left\{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right\},$$

and (2), if $r$ is the radius vector of the point,

$$n=a^2b^2c^2/\left[p\left\{p^2(r^2-a^2-b^2-c^2)+b^2c^2+c^2a^2+a^2b^2\right\}\right].$$

---

*Solution by* H. W. Curjel, B.A.; Prof. Madhavarao; *and others.*

At $(x, y, z)$ on ellipsoid $\dfrac{1}{p^2}=\dfrac{x^2}{a^4}+\dfrac{y^2}{b^4}+\dfrac{z^2}{c^4}$;

And the principal radii of curvature are given by

$$\frac{1}{\rho}=\frac{p}{d_1^2},\quad \frac{1}{\rho'}=\frac{p}{d_2^2},$$

where $d_1$, $d_2$ are the principal axes of central section of ellipsoid parallel to tangent plane at $(x, y, z)$. Equation to normal at $(x, y, z)$ is

$$(\xi-x)\Big/\frac{x}{a^2}=(\eta-y)\Big/\frac{y}{b^2}=(\zeta-z)\Big/\frac{z}{c^2};$$

therefore

$$2n=(\xi-x)\Big/\frac{px}{a^2}=(\eta-y)\Big/\frac{py}{b^2}=(\zeta-z)\Big/\frac{pz}{c^2},$$

Digitized by Google

where $(\xi, \eta, \zeta)$ is point in which normal meets ellipsoid, which gives

$$n = \frac{1}{p^2\left(\frac{x^2}{a^6} + \frac{y^2}{b^6} + \frac{z^2}{c^6}\right)} = \frac{1}{p^2 \Sigma\left(\frac{1}{a^2}\frac{x^2}{a^4}\right)}.$$

(1) Also $d_1$ and $d_2$ are given by

$$\frac{x^2}{a^4}\Big/\left(\frac{1}{a^2} - \frac{1}{d^2}\right) + \frac{y^2}{b^4}\Big/\left(\frac{1}{b^2} - \frac{1}{d^2}\right) + \frac{z^2}{c^4}\Big/\left(\frac{1}{c^2} - \frac{1}{d^2}\right) = 0;$$

therefore $$\frac{1}{d_1^2} + \frac{1}{d_2^2} = p^2 \Sigma\left\{\frac{x^2}{a^4}\left(\frac{1}{b^2} + \frac{1}{c^2}\right)\right\};$$

$$\therefore \quad \frac{1}{\rho} + \frac{1}{\rho'} + \frac{1}{n} = p^2 \Sigma\left\{\frac{x^2}{a^4}\left(\frac{1}{b^2} + \frac{1}{c^2}\right)\right\} + p^2 \Sigma\left(\frac{x^2}{a^4}\frac{1}{a^2}\right)$$

$$= p^2 \frac{1}{p^2} \Sigma \frac{1}{a^2} = p\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right).$$

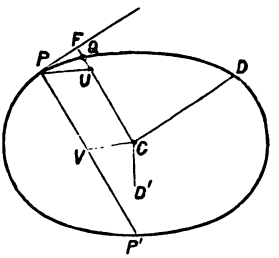
(2) Again, $$n = \frac{1}{p^2 \Sigma \frac{x^2}{a^6}} = \frac{a^2b^2c^2}{p^2 \Sigma \frac{x^2b^2c^2}{a^4}}$$

$$\Sigma \frac{x^2b^2c^2}{a^4} = \Sigma b^2c^2 \Sigma \frac{x^2}{a^4} - \Sigma\left\{\frac{x^2}{a^2}(a^2 + b^2 + c^2) - x^2\right\}$$

$$= \frac{1}{p^2} \Sigma b^2c^2 - (\Sigma a^2 - r^2) \text{ for } \Sigma \frac{x^2}{a^2} = 1;$$

hence $n$ has the value stated.

[*Otherwise*:—Take the central section through the normal chord PP′. Let CF be the perpendicular on the tangent at P to this section, meeting the ellipsoid in Q. Take V, the middle point of PP′, and draw PU parallel to VC. Let CD be the diameter parallel to tangent at P, and CD′ the diameter perpendicular to section.




$$PP' = 2PV = 2CU = \frac{2CQ^2}{CF} = 2CQ^2/p;$$

$$\therefore \quad n = CQ^2/p.$$

CQ, CD, CD′ are three perpendicular diameters; therefore

$$CD^{-2} + CD'^{-2} + CQ^{-2} = a^{-2} + b^{-2} + c^{-2};$$

but, if $\alpha$ and $\beta$ are the semi-axes of the section DCD′, then, since CD and CD′ are perpendicular,

$$CD^{-2} + CD'^{-2} = \alpha^{-2} + \beta^{-2} = (\rho \,.\, p)^{-1} + (\rho' \,.\, p)^{-1};$$

$$\therefore \quad (\rho')^{-1} + \rho^{-1} + n^{-1} = p\,(a^{-2} + b^{-2} + c^{-2}).$$

Again $\rho^{-1} + \rho^{-1} = (\alpha^{-2} + \beta^{-2}) \,.\, p = p\,(\alpha^2 + \beta^2)/\alpha^2\beta^2 = p^3\,(\alpha^2 + \beta^2)/a^2b^2c^2.$

And the plane DCD′ is conjugate to diameter CP,

Digitized by Google

$$\therefore\quad \alpha^2+\beta^2+\gamma^2 = a^2+b^2+c^2,$$

$$\therefore\quad n^{-1} = p\left\{a^{-2}+b^{-2}+c^{-2}\right\} - p^3\left(a^2+b^2+c^2-r^2\right)/a^2b^2c^2,$$

whence we obtain the stated result.]

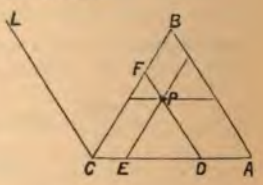
---

**5132.** (Professor Crofton, F.R.S.)—From any point P within a triangle lines are drawn parallel to the two sides, meeting the base on D, E. Show (1) (as may be done without integration) that the mean value of the triangle PDE is one-sixth of the former triangle; and find (2) the mean value of the segment DE.

---

*Solution by* Professor G. B. M. Zerr.

Let ABC be the given triangle, P the random point.




Let $CD = x$, $DP = y$, $DF = y' = cx/b$. The limits of $x$ are 0 and $b$; of $y$, 0 and $y'$.

$$DE = by/c;$$

hence area $PED = \frac{1}{2}\left[(by^2\sin A)/c\right]$,

and the required average is

$$(1)\quad \Delta = \frac{b\sin A}{2c}\int_0^b\int_0^{y'} y^2\,dx\,dy \Big/ \left\{\int_0^b\int_0^{y'} dx\,dy\right\}$$

$$= \frac{\sin A}{c^2}\int_0^b\int_0^{y'} y^2\,dx\,dy = \tfrac{1}{12}bc\sin A = \tfrac{1}{6}\text{ (area of triangle)};$$

$$(2)\quad \Delta_1 = \frac{b}{c}\int_0^b\int_0^{y'} y\,dx\,dy \Big/ \left\{\int_0^b\int_0^{y'} dx\,dy\right\} = \frac{2}{c^2}\int_0^b\int_0^{y'} y\,dx\,dy = \tfrac{1}{3}b.$$

[*Otherwise*:—Since lines through P, parallel to the sides, divide the triangle into six parts, all of which are equally likely to be equal, $PDE = \frac{1}{6}ABC$; and, since CA is divided into three parts, all of which are equally likely to be equal, $DE = \frac{1}{3}CA$.]

---

**11303.** (Professor de Bózóky.)—Combien de solutions distinctes admet l'équation $x+y+z = n+2$, les indéterminées $x$, $y$, $z$ ayant des valeurs égales ou inégales choisies parmi les nombres 1, 2, 3, ... $n$?

---

*Solution by* H. W. Curjel, B.A.; A. D. Roberts, B.A.; *and others.*

If $x = r$, $y$ may be any number from 1 to $(n-r+1)$.

Now $x$ may have any value from 1 to $n$; therefore number of solutions

$$= \sum_{r=1}^{r=n}(n-r+1) = n+n-1+n-2+\ldots+2+1 = \tfrac{1}{2}n(n+1).$$

Digitized by Google

**11437.** (Professor SYLVESTER.)—A, B, C ... K, L, M are integers; the ratios B/A, C/B, D/C ...[diminish continually towards unity as their limit. Prove that if $B-A < A^{\frac{1}{2}}$ the series must be an infinite arithmetical progression, or will become such on lopping off a sufficient number of the terms at the beginning.

*Solution by* H. W. CURJEL, B.A.

Let $B = A+a$ and $C = B+b$. Then shall $b$ be not greater than $a$, for if $b$ is $> a$, let $b = a+x$. Now $B/A > C/B$,

$$\frac{A+a}{A} > \frac{B+b}{B}; \quad \text{therefore} \quad \frac{a}{A} > \frac{a+x}{A+a},$$

whence $a > A^{\frac{1}{2}}$, *i.e.*, $B-A > A^{\frac{1}{2}}$,

which is contrary to hypothesis. Hence $b$ or $C-B$ is not greater than $a < A^{\frac{1}{2}} < B^{\frac{1}{2}}$. Similarly it may be shown that $D-C$ is not greater than $C-B$, and so on that each term of the series $B-A$, $C-B$, $D-C$, &c. is equal to or less than the preceding one, and is always positive, since B/A, C/B. D/C are greater than unity, and they are always integral; therefore after some number of terms they must all be equal, since they can only have $a$ values at the most; therefore A, B, C ... K, L, M form an infinite arithmetical progression, or will do so after a sufficient number of terms have been lopped off from the beginning.

---

**11388.** (LET DOWN TO HIS OWN INCHES BY NON-ŒDIPUS.)—

In ninety-two and ninety-three
Twelve balls of ours are fixed to be:
Ten Girton maids, Urania's pets,
We bid, in unnamed quintuplets,
To each to come,
Five left at home
(*The alternate sets*).
The beauties don't deny, &c.

*Solution by the* Rev. T. P. KIRKMAN, M.A.

Thus qualified by the italics the answer is easy. A dances six times, along with six different fours. Of such fours, there are

$$9 \cdot 8 \cdot 7 \cdot 6 \; : \; 4 \cdot 3 \cdot 2 \cdot 1 = 126.$$

Out of these 126 A can choose six fours in

$$N' = 126! : 6!\,120! = 4925156775 \text{ different ways.}$$

This N′ exactly solves the qualified question; for all the ladies get six dances, if the alternate sets are those last left at home. But the six fives led by A can mostly be completed into a correct twelve in many ways, so that N′ is far below half the N required.

The beauties may perhaps be graciously content to see this smaller though thus far correct number N′ of dozens of fives all written out on their invitation, and to wait a week or two for the odd tens of thousands of millions completing their demand for "each separate twelve," which the bachelors are bound to find. This N has to be made up by skill above mine yet. I am only 87 years old.

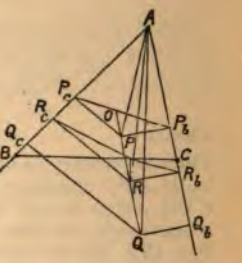


Digitized by Google

**11454.** (Professor NEUBERG.)—Soient P, Q deux points inverses par rapport au cercle circonscrit au triangle ABC ; $P_a$, $P_b$, $P_c$ les projections de P sur les côtés BC, CA, AB ; $Q_a$, $Q_b$, $Q_c$ celles de Q. La droite PQ rencontre la circonférence ABC aux points R, S. Démontrer que les côtés correspondants des triangles $P_aP_bP_c$, $Q_aQ_bQ_c$ font des angles égaux, en sens contraire, avec la droite de SIMSON du point R.

*Solution by* R. F. DAVIS, M.A.;
J. H. GRACE, M.A.; *and others.*

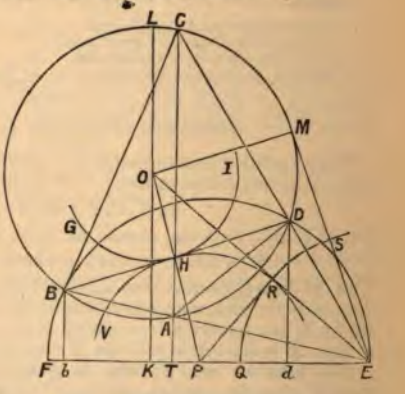
Since P, $P_b$, A, $P_c$ are obviously concyclic, $P_bP_c$ makes with AB an angle equal to the complement of PAC. Now, since $OP \cdot OQ = OR^2$, PR and RQ subtend equal angles at A (or at any point of the circum-circle of ABC); therefore the angles PAC, RAC, QAC are in A.P., and so are their equals, the angles which $P_bP_c$, $R_bR_c$, $Q_bQ_c$ make with AB. But $R_bR_c$ is the SIMSON line of R; &c.




**11420.** (The EDITOR.)—Inscribe in a circle a quadrilateral ABCD, having the diagonal AC given in direction, the diagonal BD in magnitude, and the point E in which the sides AB, CD meet.

*Solution by* D. BIDDLE; MORGAN BRIERLEY; *and others.*

Let KL, drawn through the centre of the circle, represent the direction of the diagonal AC. Draw EK perpendicular to it, and produce. On EK find P, the centre of a circle which (with radius PE) has, with the original circle, the common chord BD (given as to length). Join EB, cutting the original circle in A, and through D draw EC; also join AD, BC. ABCD is the required quadrilateral.

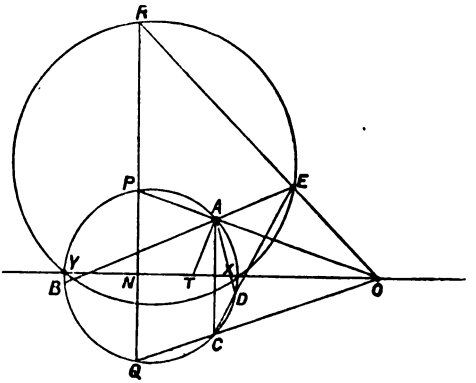



Draw B$b$, D$d$ perpendicular to EK; and, EM being the tangent to the circle, let EB : E$b$ = EM : $z$, and ED : E$d$ = EM : $u$. Then, since $EM^2 = EA \cdot EB = ED \cdot EC$, we have $Eb \cdot ET = z^2$ and $Ed \cdot ET = u^2$, whence $z^2 : u^2 = Eb : Ed = (EM \cdot Eb/EB)^2 : (EM \cdot Ed/ED)^2$; therefore $EB^2 : ED^2 = Eb : Ed$, which can only be the case when $b$, $d$ are on the

Digitized by Google

diameter (drawn from E) of the circle drawn through E, D, B, which results in $Eb : EB = EB : EF$, and $Ed : ED = ED : EF$.

P is also the centre of VHR, which touches GHI, of radius OH (known), and cuts orthogonally QRS of radius EQ ($= \frac{1}{2}$BD). This is a solved problem, equally with that adopted above.




[Mr. Davis solves the problem as follows:—Place within the given circle a chord PNQ parallel to the given direction of AC and equal to the given length of BD. Let XY be the diameter bisecting PQ at right angles in N. Let PQ meet the circle described about EXY in R; and RE, XY intersect in O. Join OP, OQ, intersecting the given circle at A, C respectively, AC being obviously parallel to PQ. Also join EA, EC, meeting the given circle in B, C respectively. Then ABCD is the quadrilateral required. For, if T be taken upon XY, such that OT . ON = OX . OY, then OT . ON also = OA . OP and OE . OR. Hence A, C, E all lie on the circle described on OT as diameter. Therefore the angle BAD = AED + ADE = AEC + ABC = AOC + APC = PCQ, and the chords BD, PQ are equal.]

---

**11475.** (D. Biddle.)—In an airless region, a marble falls from rest on to an inclined plane of indefinite extent. The elasticity of the marble and of the plane being perfect, and the force of gravity $g$ being regarded throughout as practically identical, prove that ($t$ being the time occupied before the first impact) recurring impacts will take place at intervals of $2t$, and at points which are the projections, on the plane, of the positions which would be reached at the respective intervals if the marble were falling free; progress, in fact, being made at the same rate as if the marble rolled down the plane from the projection on it of the original point of rest.

---

*Solution by* Rev. T. R. Terry, M.A.

Let marble start from A, strike plane at B, and let C be the projection of A on the plane, and let $\alpha = ABC$.

First consider the motion perpendicular to the plane. The acceleration is constant (viz., $g \sin \alpha$), and the elasticity is perfect; therefore the

Digitized by Google

velocity V perpendicular to the plane is the same just after each rebound. Therefore the time between successive impacts equals twice time in which V is generated by $g \sin \alpha$, $= 2t$.

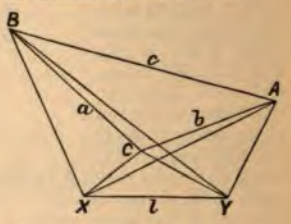
Next for motion along the plane. Since this motion is not affected by motion perpendicular to the plane, the motion along the plane is the same (1) for the bouncing marble, (2) for a marble falling freely from A, (3) for a marble slipping down the plane from C.

---

**11440.** (Professor MORLEY, M.A.)—Given a triangle *abc*, show that there are points $x$, $y$ such that the triangles $axy$, $ybx$, $xyc$, $abc$ are similar; and construct the points.

---

*Solution by* Professors SCHOUTE, BHATTHACHARYA, *and others.*




If, on the segment XY of a given line $l$, we describe three triangles AXY, YBX, XYC directly similar to a given triangle $\alpha\beta\gamma$, the triangle ABC is also directly similar to $\alpha\beta\gamma$, and $l$ is its line of LEMOINE. This new theorem, proved directly, shows that to every given triangle ABC corresponds one couple of points (X, Y) of the indicated character, and leads to a construction of these points.

After having constructed the point of LEMOINE of ABC (*i.e.*, the points whose distances to BC, CA, AB are proportional to the length $a$, $b$, $c$ of these sides) and the polar of this point with reference to the circum-circle of ABC (*i.e.*, the line of LEMOINE of ABC), the points X and Y can be found by the equality of the angles BXY, XYA, BCA.

*Proof of the Theorem.*—Similitude of triangles gives

$$BX : XY = XY : YA, \quad \angle BXY = \angle XYA,$$
$$CX : XY = XY : YB, \quad \angle CXY = \angle XYB.$$

So we find $BX : CX = BY : AY, \quad \angle BXC = \angle BYA.$

This proves the similitude of the triangles BXC and BYA, and in its turn this leads to $BX : BY = BC : BA,$

which relation, with another of the same kind, includes the similitude of ABC and $\alpha\beta\gamma$.

[According to a theorem of Professor NEUBERG'S, the locus of the third vertex of the triangles described on XY and corresponding with $\alpha\beta\gamma$ in Brocardian angle is a circle, the centre of which is at equal distances of X and Y. This proves that the projection of the circum-centre of ABC on $l$ is the mid-point of XY.]

In order to prove that $l$ is the line of LEMOINE with respect to ABC, we introduce the distances $u$, $v$, $w$ of A, B, C to $l$, and make use of the

Digitized by Google

relations $$\frac{u}{XY} = \frac{h_a}{a}, \quad \frac{v}{XY} = \frac{h_b}{b}, \quad \frac{w}{XY} = \frac{h_c}{c},$$

where $h_a$, $h_b$, $h_c$ indicate the perpendiculars of triangle ABC. They give immediately $a^2u = b^2v = c^2w$, the indicated result.

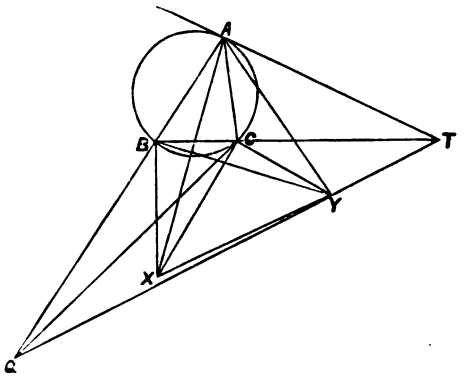
Putting $a^4 + b^4 + c^4 = p^4$, $b^2c^2 + c^2a^2 + a^2b^2 = n^4$; $XY^2 = \frac{a^2b^2c^2}{p^4 - n^4}$.

When V and W denote the isodynamic centres of ABC, we have

$$VW^2 = \frac{3a^2b^2c^2}{p^4 - n^4}; \quad \text{therefore} \quad XY = \tfrac{1}{3}\sqrt{3}\, VW.$$

[Mr. Davis solves the problem thus:—

Let ABC be a triangle having its angles A < B < C. Let the tangents to the circum-circle at A, C meet BC, AB produced in T, Q respectively. Join TQ; and let the circum-circle of ACT intersect it again in Y. Then, if upon AY there be described a triangle AXY directly similar to ABC, the triangles YBX, XYC shall be also directly similar to ABC.




Since A, C, Y, T are concyclic, the angle CYQ = CAT = ABC, and therefore Q, B, C, Y are also concyclic.

Thus AYX = C, by construction;

$$AYC = ATC = C - B, \quad CYX = B;$$
$$BYC = BQC = B - A, \quad BYX = A.$$

Again, since the triangles XAY, BAC are similar, the angle BAX = CAY, and AB : AX = AC : AY. Hence the triangles ABX, ACY are similar, and angle AXB = AYC = C − B. But AXY = B (construction); therefore BXY = C, and the triangles YBX, ABC are similar.

Finally, the angle XBC = ABY and BX : BC = YB : AB.

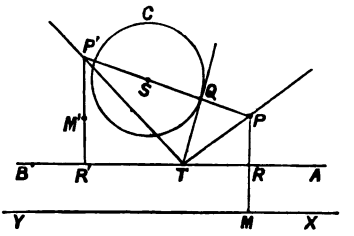
Therefore angle BXC = AYB = C − A.

Hence CXY = A, and triangles XYC, ABC are similar.]

Digitized by Google

**11408.** (Professor GENESE, M.A.)—Given a cycle C (*i.e.*, a circle with a definite sense of description), and a straight line AB. Prove that the bisector of the angle which any tangent to C makes with AB touches a parabola; and that the bisector of the angle the tangent makes with BA touches a different parabola; further, the parabolæ and the cycle have AB as common chord.

*Solution by* R. F. DAVIS, M.A.; J. W. K. CLARKE; *and others.*




Let S be the centre of the cycle C, the tangent at any point Q of which meets AB in T, S lying within the angle QTB. Produce SQ to meet the bisector of the angle QTA in P; draw PR perpendicular to AB.

Then PQ = PR, and if PR be produced to M so that RM = SQ, SP = PM. Hence PT touches, at P, a parabola whose focus is S and whose directrix XY is a line parallel to AB at a distance from it equal to the radius of C, upon the opposite side to S. Similarly, if TP′ bisect the angle QTB, the envelope of TP′ is a parabola, having its focus at S, and its directrix X′Y′ parallel to AB at a distance from it equal to the radius of C, upon the same side as S.

It can be easily seen that the points of intersection (real or imaginary) of AB with C lie on each parabola: for the distances of either of such points from S and XY (or X′Y′) are equal.

This question might very well have been suggested by reciprocation. Let O be a fixed point, and P any point upon a fixed circle, centre S. Then, if Q, Q′ be the points in which the bisectors of the angle OSP meet OP, the loci of Q, Q′ are circles having, with the given circle, their centre of similitude at O. Reciprocating with respect to the point S, we get the required property.

---

**11461.** (Col. DE LAND.)—A engaged in business which paid him a profit at the rate of 40 per cent. per annum. At the end of three months he added his profits to his business, and took in B as a partner, and then found their profits to be at the rate of 60 per cent. per annum. At the end of four months after B entered the firm they added their profits to the business, and found that A's capital was to B's as 11 to 7. They then took in C as a partner and found their losses to be at the rate of 20 per cent. per annum. At the end of six months after C entered the firm the losses were adjusted, and it was found that B's capital was to C's as 7 to 4. D now proposed to enter the firm, reorganize it, leave the capital stock the same, each partner to own an equal share, D to put in $5940, and A, B, and C to each put in or take out capital until the shares were equalized. Required (1) the capital each of the partners began with, and (2) the amount of capital each must put in or take out to equalize the shares.

Digitized by Google

*Solution by* D. Biddle; T. Savage, M.A.; *and others.*

(1) Taking the several transactions in succession, we have

$$[\{A(1{\cdot}1)+B\}(1{\cdot}2)+C](\cdot 9) = 5940\times 3 = 17{,}820 \ldots\ldots (\alpha);$$

$$A(1{\cdot}1\times 1{\cdot}2) : B(1{\cdot}2) = 11 : 7, \text{ whence } B = A(\cdot 7) \ldots\ldots (\beta);$$

$$B(1{\cdot}2\times\cdot 9) : C(\cdot 9) = 7 : 4, \text{ whence } C = B(\cdot 6'857142') = A(\cdot 48) \ldots (\gamma).$$

From $(\alpha)$, $A(1{\cdot}188+{\cdot}756+{\cdot}432) = 17{,}820 \ldots\ldots (\delta)$;

therefore $A = 7500$, $B = 5250$, $C = 3600$.

(2) From $(\delta)$ we get the following capitals of each of the three partners just before D joins:—

$$A\text{'s} = 7500\times 1{\cdot}188 = 8910, \quad B\text{'s} = 7500\times{\cdot}756 = 5670,$$

$$C\text{'s} = 7500\times{\cdot}432 = 3240.$$

The difference between these values and 5940 gives the amounts to be added or deducted; 2970, to be deducted by A; 270, to be added by B; 2700, to be added by C.

---

**11470.** (E. M. Langley, M.A.)—In a given triangle ABC an inversely similar triangle is inscribed with its vertices P, Q, R on BC, CA, AB respectively. If P = A, show that the median of PQR through P passes through a fixed point on the symmedian of ABC through A.

---

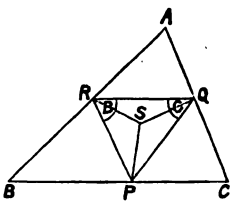
*Solution by* R. F. Davis, M.A.; I. Arnold; *and others.*

Since angle PQR = C, QR touches at Q the circum-circle of CPQ. Similarly it touches at R the circum-circle of BRP. Let these two circles intersect again at S, their common chord PS bisecting their common tangent QR. Then angle

$$QSR = B + C = \text{supplement of } A;$$

and A, Q, R, S are concyclic. Therefore angle SAR (or SAB) = SQR = SCA.

Similarly SAC = SBA.




The point S is therefore altogether fixed. It is, in fact, the vertex of Brocard's First Triangle, lying on the symmedian through A; or it may be regarded as the focus of the parabola touching AB, AC at B, C.

---

**11446.** (Professor Gob.)—Soient O le centre, et F, F' les foyers d'une ellipse. On prolonge un rayon quelconque OM de MM' = OM, et l'on mène les tangentes M'P, M'Q. P, Q étant les points de contact, démontrer la relation $4M'P \cdot M'Q = 3M'F \cdot M'F'$.

Digitized by Google

*Solution by* R. F. Davis, M.A.; J. H. Grace; *and others.*

If T be any point external to an ellipse (centre O, and foci F, F′) from which tangents TP, TQ are drawn to the curve, V the mid-point of PQ lying on OT; then, by a known theorem,

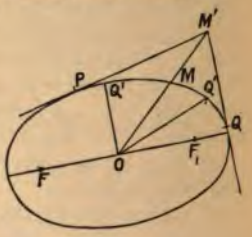
$$TP \cdot TQ : TF \cdot TF' = VT : OT.$$

When T coincides with the point M′ of the question,

$$OT = 2OM, \quad \text{and} \quad OV = \tfrac{1}{2}OM,$$

for $OV \cdot OT = OM^2$.

Hence $VT : OT = 3 : 4$; substituting, &c.




---

**11460.** (W. S. B. Woolhouse, F.R.A.S.)—Sets of Magic Squares, of specified dimensions, may be constructed and magically associated in the compartments of a square of larger dimensions such that the numbers contained in every square of the specified dimensions that can be taken therefrom shall constitute a perfect Magic Square. Show how to form Magic Squares magically associated.

---

*Solution by the* Proposer.

In what follows, the subject is principally directed to squares of *five*, having *twenty-five* compartments, which, considered alone, are somewhat interesting. A magic square, of five, is a square having 25 compartments occupied by the consecutive numbers 1 to 25, so arranged that the separate summation of each horizontal band, vertical column, and each diagonal, shall give the same amount. We proceed at once to the process of construction.

1. The first step is to make arbitrary arrangements, A and B, of the two sets of five figures, 0 1 2 3 4, 1 2 3 4 5; the same in each case to contain all the five figures, but placed in any order whatever. For example, take the following:—

A ..... 3 4 0 1 2. B ..... 2 1 5 4 3.

2. The second step is to permute these separately into clusters, thus:

| A | B |
|---|---|
| 3 4 0 1 2 | 2 1 5 4 3 |
| 0 1 2 3 4 | 4 3 2 1 5 |
| 2 3 4 0 1 | 1 5 4 3 2 |
| 4 0 1 2 3 | 3 2 1 5 4 |
| 1 2 3 4 0 | 5 4 3 2 1 |

| | | | | |
|---|---|---|---|---|
| 17 | 21 | 5 | 9 | 13 |
| 4 | 8 | 12 | 16 | 25 |
| 11 | 20 | 24 | 3 | 7 |
| 23 | 2 | 6 | 15 | 19 |
| 10 | 14 | 18 | 22 | 1 |

Digitized by Google

In permuting A, every new line begins with the last three figures of the line preceding, and the other two figures follow on.

In permuting B, every new line begins with the last two figures of the line preceding, and the other three figures follow on.

An examination of each of the clusters of figures A and B will show that each line, column, and diagonal, contains all the five figures, only differently placed. A due consideration of this fact will explain the fundamental principle of the construction.

3. The third step is to prepare the compartments of the square of *five*, and to fill in the several numbers in their due order, the same being found by taking five times each figure in the cluster A, and adding thereto the corresponding figure contained in the cluster B. Thus the initial magic square is formed.

4. The fourth and last step is to prepare the compartments of a square of ten, and then to insert in each of its four quarters the numbers contained in the initial magic square. Thus we finally obtain the

MAGIC SQUARES MAGICALLY ASSOCIATED.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 17 | 21 | 5 | 9 | 13 | 17 | 21 | 5 | 9 | 13 |
| 4 | 8 | 12 | 16 | 25 | 4 | 8 | 12 | 16 | 25 |
| 11 | 20 | 24 | 3 | 7 | 11 | 20 | 24 | 3 | 7 |
| 23 | 2 | 6 | 15 | 19 | 23 | 2 | 6 | 15 | 19 |
| 10 | 14 | 18 | 22 | 1 | 10 | 14 | 18 | 22 | 1 |
| 17 | 21 | 5 | 9 | 13 | 17 | 21 | 5 | 9 | 13 |
| 4 | 8 | 12 | 16 | 25 | 4 | 8 | 12 | 16 | 25 |
| 11 | 20 | 24 | 3 | 7 | 11 | 20 | 24 | 3 | 7 |
| 23 | 2 | 6 | 15 | 19 | 23 | 2 | 6 | 15 | 19 |
| 10 | 14 | 18 | 22 | 1 | 10 | 14 | 18 | 22 | 1 |

Any square, having 25 compartments, that can be taken within the range of the above table of family compartments, will contain the set of numbers 1 to 25, so placed as to form a complete magic square.

The selection of a square is practically facilitated by having a piece of cardboard with a square piece cut out just sufficient to exhibit the required square. Twenty-five different magic squares are thus exhibited; and by mere inspection one may be easily chosen such that any given number may occupy any stated compartment.

If a wall were papered with the table, taken as a pattern, the "magic squares" might be taken upon it *ad libitum*.



Digitized by Google

By varying the initial disposition of the figures in A and B and operating in the same way, other systems may be constructed having the same remarkable properties.

As a further illustration, the following is an example of magic squares of seven magically associated in the same manner :—

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 9 | 17 | 25 | 33 | 41 | 49 | 1 | 9 | 17 | 25 | 33 | 41 | 49 |
| 32 | 40 | 48 | 7 | 8 | 16 | 24 | 32 | 40 | 48 | 7 | 8 | 16 | 24 |
| 14 | 15 | 23 | 31 | 39 | 47 | 6 | 14 | 15 | 23 | 31 | 39 | 47 | 6 |
| 38 | 46 | 5 | 13 | 21 | 22 | 30 | 38 | 46 | 5 | 13 | 21 | 22 | 30 |
| 20 | 28 | 29 | 37 | 45 | 4 | 12 | 20 | 28 | 29 | 37 | 45 | 4 | 12 |
| 44 | 3 | 11 | 19 | 27 | 35 | 36 | 44 | 3 | 11 | 19 | 27 | 35 | 36 |
| 26 | 34 | 42 | 43 | 2 | 10 | 18 | 26 | 34 | 42 | 43 | 2 | 10 | 18 |
| 1 | 9 | 17 | 25 | 33 | 41 | 49 | 1 | 9 | 17 | 25 | 33 | 41 | 49 |
| 32 | 40 | 48 | 7 | 8 | 16 | 24 | 32 | 40 | 48 | 7 | 8 | 16 | 24 |
| 14 | 15 | 23 | 31 | 39 | 47 | 6 | 14 | 15 | 23 | 31 | 39 | 47 | 6 |
| 38 | 46 | 5 | 13 | 21 | 22 | 30 | 38 | 46 | 5 | 13 | 21 | 22 | 30 |
| 20 | 28 | 29 | 37 | 45 | 4 | 12 | 20 | 28 | 29 | 37 | 45 | 4 | 12 |
| 44 | 3 | 11 | 19 | 27 | 35 | 36 | 44 | 3 | 11 | 19 | 27 | 35 | 36 |
| 26 | 34 | 42 | 43 | 2 | 10 | 18 | 26 | 34 | 42 | 43 | 2 | 10 | 18 |

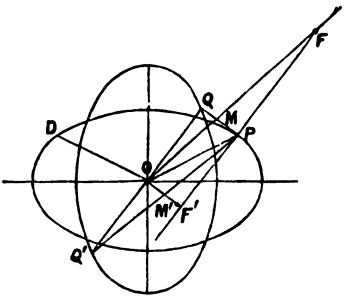
This table of family compartments exhibits forty-nine different magic squares of seven.

---

**11447.** (Professor LAISANT.)—Deux ellipses égales et de même centre O ont leurs grands axes perpendiculaires ; P est un point quelconque de la première, Q et Q′ les points de la seconde où les tangentes sont perpendiculaires à OP, et M, M′ les points milieux de PQ, PQ′ ; $a$ et $b$ étant les demi-axes des deux ellipses, démontrer que OMPM′ est un parallélogramme dont les côtés sont $\frac{1}{2}(a+b)$ et $\frac{1}{2}(a-b)$, et dont les bissectrices des angles ont les directions des axes de deux courbes.

Digitized by Google

*Solution by* R. F. Davis, M.A.; J. H. Grace, M.A.; *and others.*




Let D be an extremity of the diameter of the first ellipse which is conjugate to OP, so that the tangent at D is parallel to OP. It follows that the points Q, Q′ on the second ellipse are to be obtained by drawing the diameter QOQ′ (of the second ellipse) perpendicular to OD; OQ will be then equal to OD and parallel to the normal at P. The points O, M, M′ bisecting the sides of the triangle PQQ′, OMPM′ is a parallelogram. Produce OM, OM′ to double their lengths in the points F, F′. Then F, F′ lie on the parallel through P to MM′ (or OQ), *i.e.*, the normal at P. Also

$$PF = PF' = MM' = OQ = OD.$$

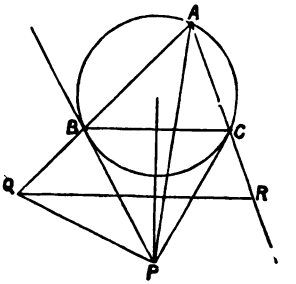
Hence, by a well known theorem, $OF = a+b$ and $OF' = a-b$.

---

**11471.** (Rev. G. Richardson.)—ABC is a triangle, A′ the mid-point of BC, and P the pole of BC, with regard to the circumcircle; through the mid-point of A′P a line is drawn parallel to BC, meeting AB, AC produced in Q, R; then A, P, Q, R are cyclic.

---

*Solution by* J. H. Grace; R. F. Davis, M.A.; *and others.*




Since QR bisects both PB, PC, it is the radical axis of the circumcircle of ABC and a point circle at P.

Hence $QP^2 = QA \cdot QB$;

and the angle

$$QPA = QBP = ACB = ARQ.$$

---

**11463.** (Artemas Martin, LL.D.)—A farmer commenced sheep-raising with a ewe lamb which had twins when one year old, one being a ewe. All the female progeny produce in this way. He sells all the sheep to the butcher when they are 4 years old, and the ewes have no lambs the last year of their lives. Thus each ewe produces three pairs of

Digitized by Google

twins, one of each of which is a ewe. If none die, what will be the number in the flock at the end of 10 years, and how many will be sold the last year?

---

*Solution by* Rev. T. R. Terry, M.A.; Prof. Sarkar; *and others.*

The statement may be simplified by adding a male sheep at first, so that at the end of the first year there are one pair of sheep one year old, and another pair just born. This additional male sheep will be sold when four years old, and will not affect the numbers afterwards. Since the number of pairs of sheep born at the end of any year is equal to the sum of the numbers of pairs born at the end of the three previous years, the numbers born at the end of 1, 2, ... 10 years are 1, 2, 4, 7, 13, 24, 44, 81, 149, 274 respectively. Therefore at the end of the tenth year 24 pairs (*i.e.*, 48 sheep) go to the butcher, and the flock consists of

$$2\left[44+81+149+274\right] = 1096 \text{ sheep.}$$

---

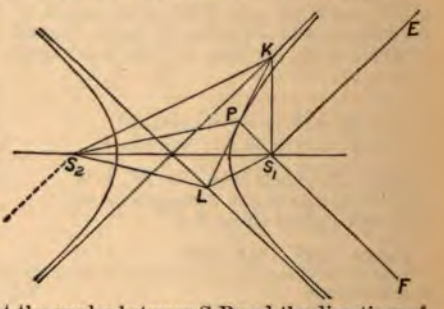
**11431.** (Rev. G. Richardson.)—Prove that the two points where the tangent to a hyperbola meets the asymptotes are concyclic with the two foci.

---

*Solution by* T. W. K. Clarke; H. W. Curjel, B.A.; *and others.*

Let the tangent at P cut the asymptotes in KL. Through the focus $S_1$ draw $S_1E$, $S_1F$ parallel to the asymptotes.

Now pairs of tangents from any point subtend equal angles at a focus; therefore $S_1K$, $S_1L$ bisect the angles $PS_1E$, $PS_1F$, respectively; therefore $\angle KSL + \frac{1}{2}$ angle between asymptotes = 2 right angles.




Similarly, $S_2K$, $S_2L$ bisect the angles between $S_2P$ and the directions of the asymptotes; therefore $\angle KS_2L = \frac{1}{2}$ angle between asymptotes = supplement angle $KS_1L$.

Therefore $S_1LS_2K$ are cyclic.

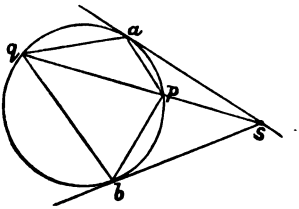
---

**11455.** (Professor Mannheim.)—On donne une circonférence de cercle et les tangentes *sa*, *sb* à cette courbe. Du point *s*, on mène une droite arbitraire qui rencontre la circonférence en *p* et en *q*; démontrer que les

Digitized by Google

distances des points de contact $a$, $b$, aux points $p$, $q$ sont proportionelles, c'est-à-dire que $\frac{ap}{aq} = \frac{bp}{bq}$.

*Solution by* R. F. Davis, M.A.; H. W. Curjel, B.A.; *and others.*




By Euclid III., 32, the triangles $sap$, $sqa$ are similar, and

$$ap : aq = sp : sa.$$

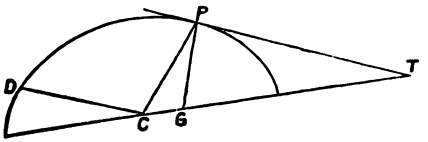
Similarly, $bp : bq = sp : sb$.

But $sa = sb$; hence, &c.

---

**11444.** (Professor Zerr.)—Prove that the intercept on the axis of abscissae made by the tangent and normal at any point on an ellipse varies as $\rho^3/x$, where $\rho$ is the radius of curvature and $x$ the abscissa of the point.

*Solution by* J. H. Grace, M.A.; M. Brierley; *and others.*




Let P be the point on the conic, C the centre and T, G the points where the tangent and normal at P meet the axis, CD the diameter conjugate to CP. We have $CT = \frac{a^2}{x}$, $CG = e^2x$, $\therefore$ $GT = \frac{a^2}{x} - e^2x = \frac{a^2 - e^2x^2}{x} = \frac{CD^2}{x}$;

but $\rho = \frac{CD^2}{ab}$; $\therefore$ $CD^2 \propto \rho^3$; $\therefore$ $GT = \frac{(ab\rho)^{\frac{2}{3}}}{x}$ or $\propto \frac{\rho^{\frac{2}{3}}}{x}$.

---

**11301.** (Professor Crofton, F.R.S.)—Prove that the $(r)^{\text{th}}$ differential coefficient of $\exp(e^x)$ is

$$D^r \exp(e^x) = \exp(e^x)\left\{e^x + \frac{\Delta^2 0^r}{2!}e^{2x} + \frac{\Delta^3 0^r}{3!}e^{3x} + \ldots e^{rx}\right\}.$$

*Solution by* H. J. Woodall.

$$D \exp(e^x) = \exp(e^x) \times e^x,$$

$$D^2 \exp(e^x) = \exp(e^x) \times e^{2x} + \exp(e^x) \times e^x = \exp(e^x)(e^x + e^{2x}),$$

Digitized by Google

$$D^3 \exp(e^x) = \exp(e^x)(e^x + e^{2x})e^x + \exp(e^x)(e^x + 2e^{2x}) = \exp(e^x)(e^x + 3e^{2x} + e^{3x}),$$

and so on.

Assume $$D^r \exp(e^x) = \exp(e^x)\left\{e^x + \frac{\Delta^2 0^r}{2!}e^{2x} + \&c.\right\},$$

$$D \cdot D^r \exp(e^x) = D^{r+1}\exp(e^x) = \exp(e^x)\left\{e^x + \frac{\Delta^2 0^r}{2!}e^{2x} + \&c.\right\}e^x$$

$$+ \exp(e^x)\left\{e^x + 2\frac{\Delta^2 0^r}{2!}e^{2x} + \&c.\right\}$$

$$= \exp(e^x)\left\{e^x + \frac{\Delta^2 0^{r+1}}{2!}e^{2x} + \&c. + e^{(r+1)x}\right\},$$

because $$\Delta^m 0^n = m(\Delta^{m-1}0^{n-1} + \Delta^m 0^{n-1}).$$

[Prof. CROFTON proves his theorem as follows:—

$$(xD)^r x^s = s^r x^s = x^s E^s 0^r = (xE)^s 0^r, \quad \text{where } E = 1 + \Delta;$$

$$\therefore \quad (xD)^r f(x) = f(xE)\,0^r;$$

$$\therefore (xD)^r e^{ex} = e^{xE}0^r = e^x e^{x\Delta}0^r = e^x\left(x\Delta 0^r + \frac{x^2}{1.2}\Delta^2 0^r + \frac{x^3}{1.2.3}\Delta^3 0^r + \ldots + x^r\right).$$

Hence $$D^r \exp(e^x) = \exp(e^x)\left(e^x + \frac{e^{2x}}{1.2}\Delta^2 0^r + \frac{e^{3x}}{1.2.3}\Delta^3 0^r + \ldots + e^{rx}\right).$$

For, let $e^x = y$; $\therefore \frac{d}{dx} = y\frac{d}{dy}$, $\therefore D^r \exp(e^x) = \left(y\frac{d}{dy}\right)^r e^{e^x}$.]

---

**8515.** (By Professor HUDSON, M.A.)—Find the area included between an epicycloid and a hypocycloid with the same number $n$ of coincident cusps.

---

*Solution by* H. J. WOODALL; *the* PROPOSER; *and others.*

Epicycloid is $$x = (a+b)\cos\theta - b\cos(a+b)/b\,.\,\theta,$$
$$y = (a+b)\sin\theta - b\sin(a+b)/b\,.\,\theta.$$

The element of area is $\frac{1}{2}(x\,dy - y\,dx)$; expressing this in terms of $\theta$, we get $(a+b)(a+2b)\sin^2(a\theta/b)\,d\theta$. For the Hypocycloid, change the sign of $b$.

Therefore the area required from one cusp to the next, if $a = nb$, becomes $$\int_0^{2\pi/n} 6ab\sin^2 n\theta\,d\theta;$$

and the whole area is $n$ times this, which is $6\pi ab$.

---

**6376.** (Prof. MINCHIN, M.A.)—A lake, the area of whose surface is $a$ square miles, and whose temperature is 32° F., is frozen to a depth of

Digitized by Google

$n$ inches in $d$ days. Calculate, in horse-power, the mean rate at which the work of freezing is done, using the well-known numerical values of the other quantities required.

*Solution by* H. J. Woodall.

When water freezes, there is internal and external work done. Considering internal work, we have

Body of water weighs $\frac{1}{12}\{an \times (1760 \times 3)^2\} \times \cdot 99987 \times 62{\cdot}42$ lbs.

Each pound, on freezing, gives out sufficient heat to warm 79·25 lbs. through 1° Cent., $= 1392 \times 79{\cdot}25$ ft.-lbs.;

hence internal work $= -336200an/d$ horse-power.

External work thus consists in raising atmosphere through $n(1-\frac{21789}{89887})$ inches over an area $a \times (1760 \times 3)^2$ sq. ft.

Work = area × height × pressure $= 4{\cdot}2an/d$ horse-power.

Work done by ice $= 336200an/d$ horse-power $-4{\cdot}2an/d$ horse-power.

**11423.** (R. W. D. Christie.)—Prove that the principle of "Casting out the Nines" may be extended to any number ending in 1, 3, 7, or 9.

[*Ex. gr.*: To test 973112691 for 37, we have $1+2+3=6$; $9+1+7=17$; $6+1+9=16$; $17-16=1$ (ten) and $6-16=-10$; therefore $10-10=0$; and the given number = 37M.]

*Solution by* F. R. J. Hervey; Professor Bordage; *and others.*

Let N be the dividend, $d$ the divisor, prime to 10; $10^n$ the lowest power which, divided by $d$, gives unit remainder; and $S_1, S_2, \ldots$ the sums of every $n^{\text{th}}$ digit of N, beginning with the 1st, 2nd, ... on the right. Then the remainder of N is that of $S_1+10S_2+10^2S_3+\ldots+10^{n-1}S_n$. If $1+10+\ldots+10^{n-1} \equiv 0$, (mod. $d$), (which is always true when $d$ is not divisible by 3), one power may be eliminated. Thus, if $d=37$, we have $10^3 \equiv 1$, (mod. $d$), and $N \equiv S_1+10S_2-11S_3 \equiv S_1-S_3+10(S_2-S_3)$, (mod. $d$). If $n=2m$, and $d$ divide $10^m+1$ (which it will, if prime), we have $N \equiv S_1-S_{m+1}+10(S_2-S_{m+2})+\ldots+10^{m-1}(S_m-S_{2m})$, (mod. $d$).

**11214.** (Professor Chakravarti.)—A straight cylindrical rod, of length $l$, symmetrical about its centre line, and of constant flexural rigidity B about any axis perpendicular to its central line, is acted on by a thrust R and opposing couples G in a plane perpendicular to R. Show that the greatest value of $l$ for which a straight central line is a stable configuration for the rod is given by $\frac{\pi^2}{l^2} = \frac{G^2}{4B^2} + \frac{R}{B}$.

Digitized by Google

*Solution by* D. EDWARDES, B.A.; Professor AIYAR; *and others.*

Let the origin be at one extremity of the rod, the axis of $z$ along the line joining the extremities, and let $u$, $v$, $z+w$ be the coordinates of any point on the central axis, when the rod is slightly deflected from the straight form. Then, considering the equilibrium of any portion of the rod, measured from one extremity, by resolving and taking moments, we have $G\frac{du}{dz}+B\frac{d^2v}{dz^2}+Rv=0$, $B\frac{d^2u}{dz^2}+Ru-G\frac{dv}{dz}=0$, $\tau\frac{d\theta}{dz}+G=0$, to the first order of small quantities. These equations are satisfied by

$$u = L\cos\lambda_1 z + L'\cos\lambda_2 z, \qquad v = M\sin\lambda_1 z + M'\sin\lambda_2 z,$$

where $\lambda_1$, $\lambda_2$ are the roots of $B\lambda^2-G\lambda-R=0$ ..... (1),

Since $u=0$, $v=0$ at the extremities, we have $(\lambda_1-\lambda_2)L=2n\pi$, where $n$ is any integer. Hence the least length for equilibrium with central axis curved, *i.e.*, the greatest length for straight form, is given by

$$\frac{\pi^2}{L^2}=\tfrac{1}{4}(\lambda_1-\lambda_2)^2=\frac{G^2}{4B^2}+\frac{R}{B}, \text{ by (1).}$$

---

**11378.** (Professor DE LONGCHAMPS.)—Démontrer que

$$\tfrac{1}{2}\Sigma a^2(b-c)^2(ab+ac-2bc)^2=\left\{b^2c^2+c^2a^2+a^2b^2-abc(a+b+c)\right\}^2.$$

*Solution by* H. J. WOODALL; R. KNOWLES, B.A.; *and others.*

$$\begin{aligned}\tfrac{1}{2}\Sigma a^2(b-c)^2(ab+ac-2bc)^2 &= \tfrac{1}{2}\Sigma a^2(b-c)^2\left\{b(a-c)+c(a-b)\right\}^2\\ &= \tfrac{1}{2}\left[\Sigma a^2b^2(b-c)^2(a-c)^2+2abc(b-c)(c-a)(a-b)\Sigma(ac-ab)\right.\\ &\qquad\left.+\Sigma a^2c^2(b-c)^2(a-b)^2\right]\\ &= a^2b^2(b-c)^2(a-c)^2+b^2c^2(c-a)^2(a-b)^2+c^2a^2(a-b)^2(b-c)^2\\ &= \left[a^2(b^2-bc+c^2)-abc(b+c)+b^2c^2\right]^2\\ &\qquad+abc^2\left[3b^2+3c^2-(b+c)^2-2(b^2-bc+c^2)\right]\\ &= \text{stated result.}\end{aligned}$$

---

**11213.** (Professor CAVALLIN.)—A quadric is described to pass through two non-intersecting given lines and to touch a given plane. Show that the locus of the point of contact with the plane is a straight line.

*Solution by* J. C. ST. CLAIR; H. W. CURJEL, B.A.; *and others.*

Let the given generating lines L, M meet the given plane in P, Q, and let the variable quadric surface touch the plane in T. Since through T there must pass two surface lines of opposite systems, both lying in the tangent plane, one of them must necessarily meet both L and M (which by hypothesis are of the same system) and must consequently be the line PQ. Hence T always lies on PQ.

Digitized by Google

**11480.** (Professor SYLVESTER.)—Let $H(x/a)$ mean $x/a$ when the fractional part of $x/a$ is $\frac{1}{2}$, and in the contrary case the nearest integer to $x/a$. Let the successive prime numbers 2, 3, 5, 7, 11, ... be denoted by $p_1, p_2, p_3, \ldots p_i$. Then, supposing that $p_i$ is $(2n)^{\frac{1}{2}}$ or the prime number next below it, $n - \Sigma H(n/p_i) + \Sigma H(n/p_ip_j) - \Sigma H(n/p_ip_jp_k) + \ldots$ (or, if it is a fraction, the integer part of it) represents the number of prime numbers greater than $n$, and not greater than $2n$.

Ex. $n = 47$, $p_1, p_2, p_3, \ldots p_i = 2, 3, 5, 7,$

$$47 - H(\tfrac{47}{3}) - H(\tfrac{47}{5}) - H(\tfrac{47}{7}) - H(\tfrac{47}{6}) + (\tfrac{47}{10}) + H(\tfrac{47}{14}) + H(\tfrac{47}{15})$$
$$+ H(\tfrac{47}{11}) + H(\tfrac{47}{35}) - H(\tfrac{47}{30}) - H(\tfrac{47}{42}) - H(\tfrac{47}{70}) - H(\tfrac{47}{105}) + H(\tfrac{47}{210})$$
$$= 47 - 23\tfrac{1}{2} - 16 - 9 - 7 + 8 + 5 + 3 + 3 + 2 + 1 - 2 - 1 - 1 - 0 + 0$$
$$= 69 - 59\tfrac{1}{2} = 9\tfrac{1}{2},$$

and there are 9 primes between 47 and 94, viz., 53, 59, 61, 67, 71, 73, 79, 83, 89.

---

*Solution by* H. W. CURJEL, B.A.

Let $I(x/a)$ = integral part of $x/a$, and $F(x/a)$ = fractional part of $x/a$; then $I(n/a)$, $I(2n/a)$ = number of multiples of $a$ not greater than $n$, $2n$;

but $I(2n/a) - I(n/a) = I(n/a) + 1$ or $I(n/a)$,

according as $F(x/a)$ is $>$ or $<$ $\frac{1}{2}$;

and $I(2n/a) - I(n/a) = I(n/a) + 1$, if $F(x/a) = \frac{1}{2}$.

Therefore, if $F(x/a)$ is not $= \frac{1}{2}$,

$H(n/a)$ = number of multiples of $a$ in the series $n+1, n+2, \ldots 2n$;

and, if $F(x/a) = \frac{1}{2}$, $H(n/a) + \frac{1}{2}$ = number of multiples of $a$ in the same series.

Now any number in that series must be a prime, or have a factor less than the square root of $2n$, the greatest number of the series, *i.e.*, must be a prime, or divisible by one of the primes $p_1, p_2, \ldots p_i$.

The number of numbers in this series is $n$; therefore, if no case occurs in which $F(n/a) = \frac{1}{2}$, the number of primes $> n$ and $\not> 2n = M$ say

$$= n - \Sigma H(n/p_i) + \Sigma H(n/p_ip_j) - \Sigma H(n/p_ip_jp_k) + \ldots;$$

for, in subtracting $\Sigma H(n/p_i)$, we have taken away all the numbers divisible by any of the series of primes $p_1, p_2, \ldots p_i$, and any number which has $r$ of those primes as factors has been subtracted $r$ times, and, in adding $\Sigma H(n/p_ip_j)$, we have added $r(r-1)/2!$ for that number, and so on; we have subtracted the number $r - r(r-1)/2! + r(r-1)(r-2)/3! +$ &c. times, *i.e.*, $1 - (1-1)^r$ times, that is, only once.

Therefore $M = n - \Sigma H(n/p_i) + \Sigma H(n/p_ip_j) -$ &c.,

if no case occurs where $F(n/p_ip_jp_k \ldots) = \frac{1}{2}$.

The only way in which $\frac{1}{2}$ can thus occur is when $n$ is odd.

Let $n$ be odd, and have $r$ factors out of the primes $p_1, p_2, \ldots p_i$. Hence it is obvious that the true value of M

$$= n - \left\{\Sigma H(n/p_i) + \tfrac{1}{2}\right\} + \left\{\Sigma H(n/p_ip_j) + r/2\right\}$$
$$- \left\{\Sigma H(n/p_ip_j) + \tfrac{1}{2}r(r-1)/2!\right\} + \ldots$$
$$= \text{given expression} - \tfrac{1}{2}\left\{1 - r + r(r-1)/2! - \ldots\right\}$$
$$= \text{given expression} - \tfrac{1}{2} \text{ or } -0.$$

Hence we obtain the result stated in the Question.



Digitized by Google

The result would be of the same form if $H(n/a) = \frac{1}{2} + n/a$, when $F(n/a) = \frac{1}{2}$. Also the only case in which $\frac{1}{2}$ will appear in the result is obviously when $n$ is a prime number.

---

**11490.** (Professor NEUBERG.)—On donne un angle droit XOY et un point A sur OX. Soit B un point variable sur OY. Sur la droite AB on prend le segment AM = OB. Le lieu de M est une courbe considérée par BARROW, qui a déterminé analytiquement la tangente au point M. Trouver cette tangente par des considérations cinématiques.

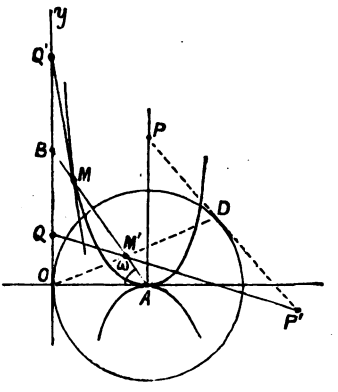
---

*Solution by* Professor DE LONGCHAMPS.

En prenant l'origine en A, AO pour axe polaire, et en posant $OA = a$, on a $\rho = a + y\omega$.

Cette courbe a été étudiée dans les *Nouvelles Annales*, 1864, p. 41, et dans le *Journal de Mathématiques Spéciales*, 1887, p. 254.

Nous voulons seulement indiquer ici comment on peut construire la tangente à cette courbe en appliquant le principe si fécond des *transversales réciproques*.




Soit M′ l'*isotomique* de M sur AB, c'est-à-dire le symétrique de M par rapport au milieu de AB. Le lieu du point M′ est une strophoïde droite ayant pour sommet le point O et dont A représente le point double.

Ayant décrit, de A comme centre, avec AO pour rayon, une circonférence A, la figure, dans laquelle on a pris P′D = DP montre comment on a construit en P′M′ la tangente à cette strophoïde.

La division AM′MB étant *isotomique*, le principe des transversales réciproques prouve que, après avoir pris BQ′ = QB, Q′M est la tangente demandée.

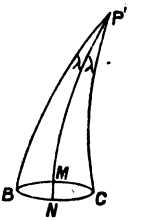
---

**11453.** (Professor LAMPE, LL.D.)—The arc BC, between two points B and C of the same parallel (latitude $\phi$), has the length of $k$ kilometres. Join B and C by the arc (BC) of the great circle through B and C, and calculate the difference BC − (BC), which is moreover the distance of the centres of BC and (BC). Put $k = 1$ kilometre, the radius of the earth being $2 \cdot 10^4/\pi$ kilometres.

Digitized by Google

*Solution by the* PROPOSER.

1. Let BNC be the arc of the parallel, BMC of the great circle, P the pole, $2\lambda$ the difference of longitudes at B and C, M the centre of arc BMC, N of BNC; then




$$PN = r(\tfrac{1}{2}\pi - \phi)\tan PM/r = \cos\lambda\cot\phi;$$

whence $MN = r\left[\tfrac{1}{2}\pi - \phi - \tan^{-1}(\cos\lambda\cot\phi)\right]$,

$$\tan\frac{MN}{r} = \frac{\cot\phi - \cos\lambda\cot\phi}{1+\cot^2\phi\cos\lambda} = \frac{\sin\phi\cos\phi(1-\cos\lambda)}{\sin^2\phi+\cos^2\phi\cos\lambda}$$

$$= \tfrac{1}{2}\lambda^2\sin\phi\cos\phi,$$

for small values of $\lambda$, whence

$$MN = r\tan^{-1}(\tfrac{1}{2}\lambda^2\sin\phi\cos\phi) = \tfrac{1}{2}r\lambda^2\sin\phi\cos\phi.$$

Now $BNC = 2r\lambda\cos\phi$ being given $= k$, we deduce

$$MN = \tfrac{1}{4}k\sin\phi\,.\,\lambda = \tfrac{1}{8}k^2/r\tan\phi.$$

2. $\sin BM/r = \cos\phi\sin\lambda$, $BM = r\sin^{-1}(\cos\phi\sin\lambda)$;

therefore $BNC - BMC = 2r\left[\lambda\cos\phi - \sin^{-1}(\cos\phi\sin\lambda)\right]$,

or
$$BNC - BMC = 2r\frac{\lambda^3}{6}\cos\phi\sin^2\phi = k^3\frac{\tan^2\phi}{24r^2}.$$

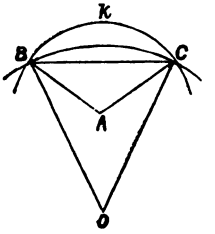
For $\phi = 45°$, we have $\tan\phi = 1$, therefore

$$MN = \frac{1}{8r} = 19{\cdot}6 \text{ mm}; \quad BNC - BMC = \frac{1}{24r^2} = 0{\cdot}00103 \text{ mm}.$$

[Prof. ZERR gives a solution thus:—

Let $BO = r$, $\angle BAC = \theta$, $\angle BOC = \mu$,

$$\pi/180 = a.$$




Then $BA = r\cos\phi = r'$, $k = ar'\theta$,

or $\theta = k/ar'$, $(BC) = ar\mu$,

chord $BC = 2r\sin\tfrac{1}{2}\mu = 2r\cos\phi\sin\tfrac{1}{2}\theta$

$$= 2r\cos\phi\sin k/2ar'$$

$$= 2r\cos\phi\sin\left[k/(2ar\cos\phi)\right],$$

$$\therefore \quad (BC) = 2ar\sin^{-1}\left\{\cos\phi\sin\left[k/(2ar\cos\phi)\right]\right\},$$

$$BC - (BC) = k - 2ar\sin^{-1}\left\{\cos\phi\sin\left[k/(2ar\cos\phi)\right]\right\}$$

$= 0$, when $\phi = 0$,

$$= 1 - (4.10^4/180)\sin^{-1}\left\{\cos\phi\sin\left[180/(4.10^4.\cos\phi)\right]\right\}.$$

From this, when $\phi$ is known, an approximate value can be found.]

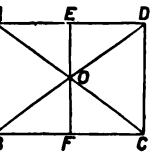
---

**11493.** (Professor SCHOUTE.)—Of a quadrangle two opposite sides are equal. Prove that the lines bisecting orthogonally the other sides coincide or intersect *without* the quadrangle.

Digitized by Google

*Solution by* H. W. CURJEL; MORGAN BRIERLEY; *and others.*

Let ABCD be a quadrilateral having AB = CD; and let the lines EO, OF, which bisect AD and BC at right angles, cut in O. Join AO, OD, OB, OC. OA = OD, and OB = OC, and AB = DC;




$\therefore$ $\angle BAO = \angle ODC$, and $\angle OBA = \angle OCD$.

But, since OA = OD, $\angle OAD = \angle ODA$;

and, since OB = OC, $\angle OBC = \angle OCB$.

If O is within the quadrilateral,

$$\angle BAD = BAO + OAD, \quad \text{and} \quad \angle ADC = ADO + ODC,$$

and $\angle ABC = ABO + OBC$, and $\angle BCD = BCO + OCD$.

Therefore, if O is within the quadrilateral,

$$\angle BAD = \angle ADC \quad \text{and} \quad \angle ABC = \angle BCD;$$

therefore AD is parallel to BC, and OE and OF are in the same straight line. That is, OE, OF coincide or intersect without the triangle.

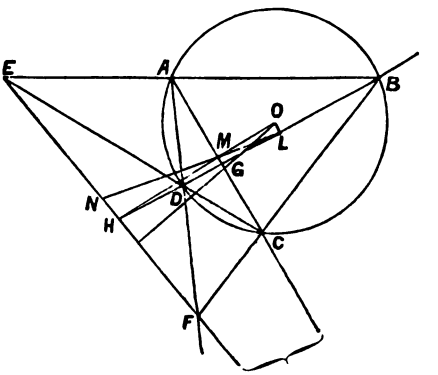
---

**11429.** (E. M. LANGLEY, M.A.)—Two circles are described, each passing through one end of the third diagonal of a cyclic quadrilateral and the middle points of the other two. Show that the third diagonal is a common tangent.

---

*Solution by* H. W. CURJEL, B.A.; M. BRIERLEY; *and others.*

Let O be the centre of circle, and let ABCD be the quadrilateral, BA, CD meeting in E, and AD, BC meeting in F, AC and BD in G, EF and BD in H, AC and EF in I.




Then the middle points L, M, N of BD, AC, EF are collinear.

H is pole of AC. Therefore OMH is a straight line, and perpendicular to AC.

G is pole of EF. Therefore OG is perpendicular to EF.

$\therefore$ $\angle HIM = \angle MOG$,

$= \angle MLG$; for O, L, G, M are concyclic.

Digitized by Google

$\therefore$ M, L, I, H are concyclic.

$\therefore$ NM . NL = NH . NI,

= $NE^2 = NF^2$ (for H, I are harmonic conjugates of E, F).

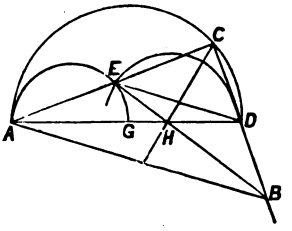
$\therefore$ ENF touches circles EML, FML.

---

**11508.** (I. Arnold.) — Construct a right-angled triangle, having given the median lines from the extremities of the hypotenuse.

*Solution by* H. W. Curjel, B.A.; Morgan Brierley; *and others.*

Let AD be one of the given medians. Bisect AD in G, and cut off from DA, DH = $\frac{1}{3}$AD. On AG and AD describe semicircles on the same side of AD. And from centre H, with radius = $\frac{1}{3}$X, where X = other given median, describe a circle cutting semicircle on AG in E.




Produce AE to meet semicircle on AD in C; and let EH and CD meet in B. Then E is the middle point of AC, and $\triangle$AEH, CHD, EHC are equal.

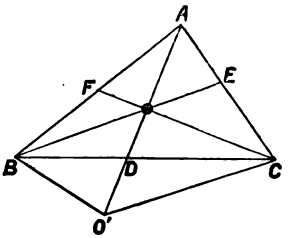
Now $\triangle EHD = \frac{1}{2}\triangle CHD = \frac{1}{3}\triangle CEH$; therefore CB = 2BD; therefore AD is a median of $\triangle$ABC, and BE is also a median of $\triangle$ABC; therefore EB = 3EH = X. And, since C is a right angle, ABC is the required triangle.

[Mr. Greenstreet and others give the following construction: — Let BE and CF be the given medians of the $\triangle$ABC (A = 90°). Then

$BO^2 + OC^2 = 2OD^2 + 2BD^2$

$= \frac{9}{10}AD^2 + 2AD^2 = \frac{29}{10}AD^2$,

$\therefore$ $AD^2 = \frac{1}{5}(BE^2 + CF^2)$.




Hence the three medians are known. Construct the $\triangle$BOO', the sides of which are two-thirds of these medians. Bisect OO' in D, and produce BD to C, so that BD = DC; and produce O'O to A, so that O'O = OA. Then ABC is the triangle required. For

$AD^2 = \frac{9}{20}(BO^2 + OC^2) = \frac{9}{20}(2OD^2 + 2BD^2) = \frac{9}{10}(\frac{1}{9}AD^2 + BD^2)$,

$\therefore$ AD = DB = DC, and A = 90°.

---

**11449.** (Professor Tissot.)—Le centre d'un cercle bi-tangent à une conique, la projection d'un point quelconque de la conique sur la corde

Digitized by Google

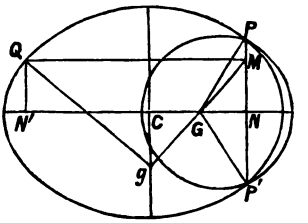
des contacts et l'intersection de la normale en ce point avec l'axe qui ne contient pas le centre du cercle, sont en ligne droite.

---

*Solution by* J. H. Grace; R. Knowles, B.A.; *and others.*

Let PP′ be the chord of contact, cutting the major axis in N. Let G be the centre of the circle.

From Q, another point on the conic, draw QM perpendicular to PP′. Join MG, and let it cut the axis minor in $g$. Let QN′ be the ordinate of Q.




We have $CG = e^2 : CN$,

$\therefore$ $CG : CN = e^2 : 1$;

$\therefore$ $CG : GN = e^2 : 1-e^2$,

$\therefore$ $Cg : MN = e^2 : 1-e^2$.

Hence, if Q be the point $(x_1, y_1)$, the equation of the normal at Q is

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2 = a^2e^2.$$

Therefore, where this normal cuts the minor axis, we have

$$\frac{y}{y_1} = -\frac{a^2-b^2}{b^2} = -\frac{a^2e^2}{a^2(1-e^2)},$$

which shows that the normal at Q passes through $g$, and therefore proves the theorem.

---

**11445.** (Professor Davis.)—If the centres of three equilateral triangles described on the three sides of a scalene triangle be joined by straight lines, prove by geometry that the triangle thus formed is equilateral.

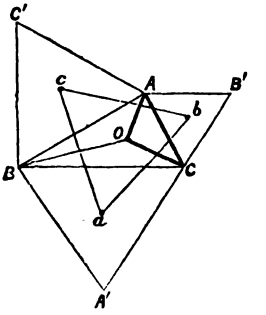
---

*Solutions by* G. E. Crawford, B.A.; Morgan Brierley; *and others.*

Circles drawn round ACB′, BCA′, ABC′, must co-intersect at O; and the angles AOB, BOC, COA are each 120°.

Now BO is the common chord of the two circles whose centres are $a$ and $c$. Therefore $ca$ is perpendicular to BO, and likewise $ab$ to OC and $bc$ to OA.

But OA, OB, OC are equally inclined to each other; hence $ab$, $bc$, $ca$ are so too. Therefore $abc$ is an equilateral triangle.



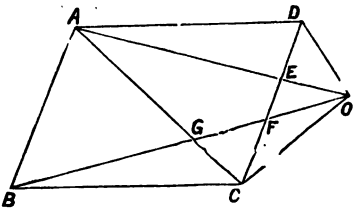

Digitized by Google

**11418.** (Professor Liégeb.)—On considère un parallélogramme ABCD et un point extérieur O, compris entre les prolongements des côtés AD, BC. On joint le point O aux sommets A, B, C, D ; soient E, F, G les points de rencontre respectifs des couples de droites OA et DC, OB et DC, OB et AC. Démontrer que le quadrilatère AEFG équivaut à la somme des trois triangles BCG, CFO, DEO.

---

*Solution by* M. Brierley ; R. F. Davis, M.A. ; *and others.*

Join OC, OD. Since AB is equal and parallel to CD, and the perpendicular from O on AB is equal to the sum of the perpendiculars from C on AB and O on CD, the triangle OAB is equal to the two triangles ABC, OCD together. From each take the triangles ABG, OEF. Whence, &c.




---

**11479.** (M. Brierley.)—If perpendiculars AD, BE, CF, from the vertices to the sides of the triangle ABC, intersect in P ; prove that

$$\frac{AB.BC}{AP.CP} = \frac{BE}{PE'} \quad \frac{AC.BC}{AP.BP} = \frac{CF}{PF'} \quad \frac{AC.AB}{CP.BP} = \frac{AD}{PD}.$$

---

*Solution by the* Rev. T. R. Terry, M.A. ; Prof. Zerr, M.A. ; *and others.*

If R be the circumradius, it is well known that

$$AP = 2R\cos A, \quad PD = 2R\cos B\cos C, \quad AD = 2R\sin B\sin C.$$

Therefore
$$\frac{AB.BC}{AP.CP} = \tan A\tan C = \frac{BE}{PE'}$$

and the other two results follow in the same way.

---

**8225.** (D. Biddle.)—A sphere 12 inches in diameter rolls down a roof which is inclined at 30° to the horizon, the eaves being 20 feet from the ground ; 10 feet in advance of the vertical line thus indicated, and 3 feet from the ground, is the upper and further edge of a horizontal bar of square section. Disregarding friction and the resistance of the air,

Digitized by Google

find how far the sphere must have rolled on the roof, from a position of rest, in order to clear the bar.

---

*Solution by* Professor ZERR.

From ROUTH's *Rigid Dynamics*, pp. 124, space passed over ($z$)

$$= \frac{5}{14} g \sin a \,.\, t^2 = \frac{5gt^2}{28};$$

also $$v = \frac{dz}{dt} = \frac{5}{7} g \sin a \,.\, t = \frac{5gt}{14};$$

therefore $$t^2 = \frac{196v^2}{25g^2}; \quad \text{therefore } z = \frac{7v^2}{5g}.$$

From the equation to the trajectory,

$$y = x \tan a - gx^2/(2v^2 \cos^2 a),$$

we have, taking the path of the centre of the sphere,

$$y = -17\tfrac{1}{2} \text{ feet}, \quad x = 10\tfrac{1}{2} \text{ feet}, \quad a = 330^\circ;$$

therefore $$v^2 = \frac{21g}{5-\sqrt{3}}; \quad \text{hence } z = \frac{147}{25-5\sqrt{3}} = 9 \text{ feet nearly.}$$

[The PROPOSER, whilst agreeing with the general method of the above solution, observes that the centre is beyond (and not directly over) the eaves when the sphere leaves the roof and begins the parabolic curve.

Hence $$x = 10\tfrac{1}{4}, \quad y = -(17 + \tfrac{1}{4}\sqrt{3}),$$

and $$v^2 = \frac{(41)^2 g}{408 - 79\sqrt{3}}; \qquad \therefore \quad z = \frac{7\,(41)^2}{2040 - 395\sqrt{3}} = 8{\cdot}84.]$$

---

**11511.** (R. W. D. CHRISTIE.)—Prove that the sum of four integral squares can in general be resolved into four others; and give a simple rule for the transformation.

*Ex. gr.*:— $$1^2 + 4^2 + 6^2 + 7^2 = 2^2 + 3^2 + 5^2 + 8^2.$$

---

*Solution by* R. F. DAVIS, M.A.

If $a$, $b$, $c$, $d$ be four positive integers, and two groups, each of four elements, be formed in the following manner,

$\frac{1}{2}(a+b+c+d)$, $\frac{1}{2}(a+b-c-d)$, $\frac{1}{2}(a-b+c-d)$, $\frac{1}{2}(a-b-c+d)$ ... (I.),

$\frac{1}{2}(-a+b+c+d)$, $\frac{1}{2}(a-b+c+d)$, $\frac{1}{2}(a+b-c+d)$, $\frac{1}{2}(a+b+c-d)$...(II.),

then the sum of the squares of the elements in either of the groups (I.) or (II.) is equal to $a^2 + b^2 + c^2 + d^2 \equiv (a, b, c, d)^2$.

If $a+b+c+d$ is even, the above transformation is directly applicable.

Digitized by Google

For, classifying the different cases, we may take, as illustrations, when $a, b, c, d$ are—

(i.) All even, $(6, 22, 38, 46)^2 = (4, 12, 28, 56)^2 = (10, 18, 34, 50)^2$.

(ii.) All odd, $(5, 17, 31, 59)^2 = (8, 20, 34, 56)^2 = (3, 25, 39, 51)^2$.

(iii.) Two even and two odd,

$$(6, 31, 44, 59)^2 = (5, 20, 33, 70)^2 = (11, 26, 39, 64)^2.$$

When $a+b+c+d$ is odd, the above transformation is not directly applicable. As, however, any three of the four given quantities are such that $a \pm b \pm c = 3m$, $m^2$ must be added to both sides, and, on transformation as before, will cancel out. Thus—

(iv.) Three odd, one even, $S = (7, 20, 29, 43)^2$;

then $S + 10^2 = 29^2 + (7, 10, 20, 43)^2 = 29^2 + (10, 13, 23, 40)^2$;

$$\therefore\ S = (13, 23, 29, 40)^2.$$

Again, $S + 8^2 = 23^2 + (8, 13, 29, 40)^2 = 23^2 + (3, 8, 24, 45)^2$;

$$\therefore\ S = (3, 23, 24, 45)^2.$$

(v.) One odd, three even, $S = (6, 11, 34, 58)^2$;

then $S + 25^2 = 34^2 + (6, 11, 25, 58)^2 = 34^2 + (8, 25, 39, 44)^2$;

$$\therefore\ S = (8, 34, 39, 44)^2.$$

Again, $S + 27^2 = 44^2 + (8, 27, 34, 39)^2 = 44^2 + (15, 20, 27, 46)^2$;

$$\therefore\ S = (6, 15, 20, 44)^2.$$

[The Proposer's solution is as follows:—Let $a^2+b^2+c^2 \equiv \alpha^2+\beta^2+\gamma^2$, where $a$, $b$, and $c$ may assume any one of the forms $3m$, $3m \pm 1$. We shall then have 10 possible varieties and no more.

For (1), $(3m_1)^2 + (3m_2)^2 + (3m_3)^2$

$$\equiv \{2m_1 + 2m_2 + m_3\}^2 + \{2m_2 + 2m_3 - m_1\}^2 + \{2m_3 + 2m_1 - m_2\}^2$$

$$\equiv \alpha^2 + \beta^2 + \gamma^2 \text{ say.}$$

$\therefore$ (2) $(3m_1+1)^2 + (3m_2+1)^2 + (3m_3+1)^2 \equiv (\alpha+1)^2 + (\beta+1)^2 + (\gamma+1)^2$;

and (3) $(3m_1-1)^2 + (3m_2-1)^2 + (3m_3-1)^2 \equiv (\alpha-1)^2 + (\beta-1)^2 + (\gamma-1)^2$;

and (4) $(3m_1)^2 + (3m_2+1)^2 + (3m_3+2)^2 \equiv \alpha^2 + (\beta+2)^2 + (\gamma+1)^2$,

since the sums of the roots are also equal.

Similarly (5) $(3m_1)^2 + (3m_2+1)^2 + (3m_3+1)^2$

$$\equiv \{2m_3 + m_1 - 2m_2\}^2 + \{2m_3 + m_2 + 1 - 2m_1\}^2 + \{2m_1 + 2m_2 + m_3 + 1\}^2$$

$$\equiv \alpha^2 + \beta^2 + \gamma^2 \text{ say.}$$

Then (6) $(3m_1)^2 + (3m_2+2)^2 + (3m_3+2)^2 \equiv \alpha^2 + (\beta-1)^2 + (\gamma+1)^2$.

Similarly (7) $(3m_1+1)^2 + (3m_2+2)^2 + (3m_3+2)^2$

$$\equiv \{2m_2 + 2m_3 + m_1 + 3\}^2 + \{2m_1 + m_2 - m_3\}^2 + \{2m_1 + m_3 - 2m_2\}^2.$$

$\therefore$ (8) $(3m_1+2)^2 + (3m_2+1)^2 + (3m_3+1)^2 \equiv (\alpha+1)^2 + (\beta-1)^2 + (\gamma-1)^2$.

But (9) $(3m_1+1)^2 + (3m_2)^2 + (3m_3)^2 \equiv$ obstinate,

and (10) $(3m_1+2)^2 + (3m_2)^2 + (3m_3)^2 =$ obstinate.

We thus see that, in eight out of the ten possible cases, we can directly resolve the sum of three squares into three others.

Of the two obstinate cases, let $d^2$ be the *fourth* square. Now, w atever form $d$ assumes, whether $3m$ or $3m \pm 1$, we can then secure, out of the



Digitized by Google

four squares, three squares falling under one or other of the above eight tractable cases.

So that it is always possible to resolve the sum of four squares into four others, and often in various ways.]

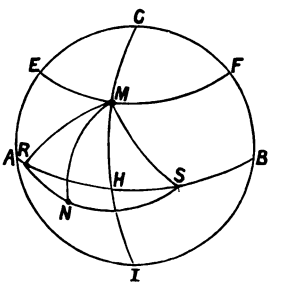
---

**7560.** (Prof. Asûtosh Mukhopadhyay, M.A.)—Two points are taken at random on the surface of a sphere, of radius $\rho$, on the opposite sides of a fixed great circle; prove that the average distance between them is $\rho\left(\pi - \frac{4}{\pi}\right)$.

---

*Solution by* Professor G. B. M. Zerr.

Let M and N be the two points taken at random on opposite sides of the fixed great circle AHB. Through M draw the meridian CHI, and also the circle EMF parallel to AHB. From M as a pole, and with a spherical radius equal to MN, describe the arc RNS, intersecting AHB in R and S.




Let the arc $MN = \rho\theta$, $HM = \rho\phi$, $RH = SH = \rho u$, and the angle $RMH = \psi$.

Then the chord $MN = 2\rho \sin\frac{1}{2}\theta$,

$$\sin\mu = \sin\theta \sin\psi \quad \ldots\ldots\ldots (1),$$

$$\cos\psi = \tan\phi \cot\theta \quad \ldots\ldots\ldots (2),$$

$$\cos\theta = \cos\phi \cos\mu \quad \ldots\ldots\ldots (3).$$

An element of surface at M is $2\pi\rho\cos\phi\,\rho d\phi$. Now, if $\theta$ is less than $\frac{1}{2}\pi$, while $\phi$ varies from 0 to $\theta$, for each position of M, N may be taken anywhere in the arc RNS, whose length is $2\psi\rho\sin\theta$. If $\theta$ is greater than $\frac{1}{2}\pi$, while $\phi$ varies from 0 to $\pi-\theta$, for each position of M, N may be taken anywhere in the arc RNS; but while $\phi$ varies from $\pi-\theta$ to $\frac{1}{2}\pi$, for each position of M, N may be taken anywhere in the circumference whose length is $2\pi\rho\sin\theta$. Hence, since the whole number of ways the two points can be taken is $4\pi^2\rho^4$,

$$\therefore\ \Delta = \frac{1}{4\pi^2\rho^4}\int_0^{\frac{1}{2}\pi}\int_0^{\theta} 2\rho\sin\tfrac{1}{2}\theta \cdot 2\psi\rho\sin\theta\,\rho d\theta \cdot 2\pi\rho\cos\phi\,\rho d\phi$$

$$+ \frac{1}{4\pi^2\rho^4}\int_{\frac{1}{2}\pi}^{\pi}\int_0^{\pi-\theta} 2\rho\sin\tfrac{1}{2}\theta \cdot 2\psi\rho\sin\theta\,\rho d\theta \cdot 2\pi\rho\cos\phi\,\rho d\phi$$

$$+ \frac{1}{4\pi^2\rho^4}\int_{\frac{1}{2}\pi}^{\pi}\int_{\pi-\theta}^{\frac{1}{2}\pi} 2\rho\sin\tfrac{1}{2}\theta \cdot 2\pi\rho\sin\theta\,\rho d\theta \cdot 2\pi\rho\cos\phi\,\rho d\phi$$

$$= \frac{2\rho}{\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\theta} \sin\theta\sin\tfrac{1}{2}\theta\,d\theta\,\psi\cos\phi\,d\phi$$

$$+ \frac{2\rho}{\pi}\left[\int_{\frac{1}{2}\pi}^{\pi}\int_0^{\pi-\theta} \psi\cos\phi\,d\phi + \int_{\pi-\theta}^{\frac{1}{2}\pi} \pi\cos\phi\,d\phi\right]\sin\theta\sin\tfrac{1}{2}\theta\,d\theta.$$

Digitized by Google

But $\int \psi \cos\phi d\phi = \psi \sin\phi - \int \sin\phi d\psi.$

From (1) $\sin\theta \cos\psi d\psi = \cos\mu d\mu$ .....(4).

From (2) and (4), $\cos\theta \sin\phi d\psi = \cos\phi \cos\mu d\mu$ ..... (5).

From (3) and (5), $\sin\phi d\psi = d\mu$;

$$\therefore \int \psi \cos\phi d\phi = \psi \sin\phi - \mu.$$

Now, observing that when $\phi = 0$, $\psi = \frac{1}{2}\pi$, and $\mu = \theta$; when $\phi = \theta$, $\psi = 0$, and $\mu = 0$; and when $\phi = \pi - \theta$, $\psi = \pi$, and $\mu = \pi$, we have

$$\Delta = \frac{2\rho}{\pi}\int_0^{\pi} \theta \sin\theta \sin\tfrac{1}{2}\theta d\theta = \frac{8\rho}{3}\left(1 - \frac{4}{3\pi}\right).$$

For the distance measured on the surface of the sphere, write $\theta$ for $2\sin\frac{1}{2}\theta$, and we have

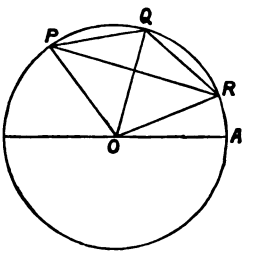
$$\Delta = \frac{\rho}{\pi}\int_0^{\pi} \theta^2 \sin\theta d\theta = \rho\left(\pi - \frac{4}{\pi}\right).$$

---

**2039.** (The late Prof. WOLSTENHOLME.)—A triangle is inscribed in a given circle whose radius is $r$: prove that, if all positions of the angular points be equally probable, the mean value of the perimeter is $120/\pi$, and that the mean value of the radius of the inscribed circle is $r(12/\pi^2 - 1)$.

*Solution by* Professor G. B. M. ZERR.

Let PQR be the triangle inscribed in the circle with centre O.




Let OA $= r$, $\angle$POA $= \theta$, $\angle$QOA $= \psi$, $\angle$ROA $= \phi$.

Then $2r\left\{\sin\frac{1}{2}(\theta-\phi) + \sin\frac{1}{2}(\theta-\psi) + \sin\frac{1}{2}(\psi-\phi)\right\} = u$,

where $u$ = perimeter.

and $r\left\{\sin\frac{1}{2}(\theta-\psi) + \sin\frac{1}{2}(\psi-\phi) - \sin\frac{1}{2}(\theta-\phi)\right\} \cot\frac{1}{4}(\theta-\phi)$

$= u_1$ = radius of the inscribed circle.

An element of arc, at P, is $r d\theta$; at Q, $r d\psi$; at R, $r d\phi$.

The limits of $\theta$ are 0 and $2\pi$; of $\phi$, 0 and $\theta$; of $\psi$, $\phi$ and $\theta$. Hence

$$(1)\quad \Delta = \frac{\int_0^{2\pi}\int_0^{\theta}\int_{\phi}^{\theta} u r d\theta r d\phi r d\psi}{\int_0^{2\pi}\int_0^{\theta}\int_{\phi}^{\theta} r d\theta r d\phi r d\psi} = \frac{3}{4\pi^3}\int_0^{2\pi}\int_0^{\theta}\int_{\phi}^{\theta} u d\theta d\phi d\psi$$

$$= \frac{3r}{2\pi^3}\int_0^{2\pi}\int_0^{\theta}\left\{(\theta-\phi)\sin\tfrac{1}{2}(\theta-\phi) + 4 - 4\cos\tfrac{1}{2}(\theta-\phi)\right\} d\theta d\phi$$

$$= \frac{3r}{\pi^3}\int_0^{2\pi}(2\theta - \theta\cos\tfrac{1}{2}\theta - 2\sin\tfrac{1}{2}\theta)\, d\theta = \frac{12r}{\pi};$$

Digitized by Google

$$(2)\quad \Delta_1 = \frac{\int_0^{2\pi}\int_0^{\theta}\int_{\phi}^{\theta} u_1\, r d\theta\, r d\phi\, r d\psi}{\int_0^{2\pi}\int_0^{\theta}\int_{\phi}^{\theta} r d\theta\, r d\phi\, r d\psi} = \frac{3}{4\pi^3}\int_0^{2\pi}\int_0^{\theta}\int_{\phi}^{\theta} u_1\, d\theta\, d\phi\, d\psi$$

$$= \frac{3r}{4\pi^3}\int_0^{2\pi}\int_0^{\theta}\left\{4 - 4\cos\tfrac{1}{2}(\theta-\phi) - (\theta-\phi)\sin\tfrac{1}{2}(\theta-\phi)\right\}\cot\tfrac{1}{4}(\theta-\phi)\, d\theta$$

$$= \frac{3r}{8\pi^3}\int_0^{2\pi}(16 - 16\cos\tfrac{1}{2}\theta - \theta^2 - 4\theta\sin\tfrac{1}{2}\theta + 16\sin^2\tfrac{1}{4}\theta)\, d\theta = r\left(\frac{12}{\pi^2} - 1\right).$$

---

**9203.** (Professor NASH, M.A.)—Prove that the two circles

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = (a\alpha + b\beta + c\gamma)(l\alpha + m\beta + n\gamma),$$
$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = (a\alpha + b\beta + c\gamma)(l'\alpha + m'\beta + n'\gamma)$$

will be orthogonal if

$$1 - \Sigma l\cos A - \Sigma l'\cos A + \Sigma ll' - \Sigma\,(mn' + m'n)\cos A = 0,$$

and therefore the condition that (1) may be orthogonal to the circumcircle is $\Sigma l\cos A = 1$.

---

*Solution by* H. J. WOODALL.

The condition is that the pole of the radical axis with respect to one circle must be the centre of the other circle, *i.e.*, the pole of the line infinity with respect to the other circle. This radical axis is

$$(l - l')\,\alpha + (m - m')\,\beta + (n - n')\,\gamma = 0\ ;$$

polar of $(\alpha', \beta', \gamma')$ with respect to (1) is

$$\alpha\left\{a\,(l\alpha' + m\beta' + n\gamma') + l\,(a\alpha' + b\beta' + c\gamma') - b\gamma' - c\beta'\right\} + \&c.\ ;$$

if these coincide, we have

$$\left\{a\,(l\alpha' + m\beta' + n\gamma') + l\,(a\alpha' + b\beta' + c\beta') - b\gamma' - c\beta'\right\}/(l - l')$$
$$= \text{symmetrical expressions}\ldots\ldots\ldots\ldots\ldots\ldots(\text{I.}),$$

$(\alpha', \beta', \gamma')$ being centre of circle (2) = pole of line infinity.

Therefore $(\alpha', \beta', \gamma')$ satisfies

$$\left\{a\,(l'\alpha' + m'\beta' + n'\gamma') + l'\,(a\alpha' + b\beta' + c\gamma') - b\gamma' - c\beta'\right\}/a = \&c.,$$

which reduces to $\left\{l'\,(a\alpha' + b\beta' + c\gamma') - b\gamma' - c\beta'\right\}/a = \&c\ldots\ldots\ldots\ldots\ldots\ldots(\text{II.})$.

These produce four equations, by which we obtain the determinant

$$\begin{Vmatrix} 2al\,(m-m') - (l-l')(bl + am - c), & (n-n')\,2al - (l-l')(cl + an - b), \\ (m-m')(am + bl - c) - (l-l')\,2mb, & (n-n')(am + bl - c) - (l-l')(cm + bn - a), \\ (m-m')(an + cl - b) - (l-l')(bn + cm - a), & (n-n')(an + cl - b) - (l-l')\,2cn, \\ abl' - a^2m' + ac, & acl' - a^2n' + ab \\ b^2l' - bc - abm', & bcl' - c^2 - abn' + \\ bcl' - b^2 - acm' + a^2, & c^2l' - bc - acn' \end{Vmatrix} = 0.$$

Digitized by Google

This is equivalent to two conditions, which may be expressed

$$\{b(l-l')-a(m-m')\}[1-\Sigma l\cos A-\Sigma l'\cos A+\Sigma ll'-\Sigma(mn'+m'n)\cos A = 0 \quad \text{.....(III.)},$$

$$\{c(l-l')-a(n-n')\}[1-\Sigma l\cos A-\Sigma l'\cos A+\Sigma ll'-\Sigma(mn'+m'n)\cos A] = 0 \quad \text{.....(IV.)};$$

but $(l-l')/a \neq (m-m')/b \neq (n-n')/c$; therefore

$$1-\Sigma l\cos A-\Sigma l'\cos A+\Sigma ll'-\Sigma(mn'+m'n)\cos A = 0.$$

If the second circle is the circumcircle $l'=m'=n'=0$; therefore condition becomes $\Sigma l\cos A = 1$.

---

**11466.** (R. Tucker, M.A.)—Parallelograms $\Omega A\Omega'D$, $\Omega B\Omega'E$, $\Omega C\Omega'F$ are formed, where $\Omega$, $\Omega'$, are the B.-points of a triangle. Prove that D, E, F lie on a circle (equal the circumcircle) which cuts the sides of ABC in points $(p, q, r)$, $(p', q', r')$, so as to make the angles

$$\Omega pA = \Omega' p'A = \omega = \&c.$$

---

*Solution by the* Proposer.

The equations are $A\Omega$, $bc\beta-a^2\gamma=0$; $A\Omega'$, $a^2\beta-bc\gamma=0$;

hence $\Omega'D$, $ca\omega_a\alpha-a^2bck\beta+b^2(a^4+a^2c^2+c^4)\gamma=0$;

$\Omega D$, $ab\omega_a\alpha+{}^2(a^4+a^2b^2+b^4)\beta-a^2bck\gamma=0$;

where $\omega_a \equiv b^2c^2-a^4$, $k \equiv a^2+b^2+c^2$.

Hence, D, $\dfrac{a\alpha}{-b^2c^2} = \dfrac{\beta}{b(c^2+a^2)} = \dfrac{\gamma}{c(a^2+b^2)}$;

E, $\dfrac{a}{a(b^2+c^2)} = \dfrac{b\beta}{-c^2a^2} = \dfrac{\gamma}{c(a^2+b^2)}$;

F, $\dfrac{a}{a(b^2+c^2)} = \dfrac{\beta}{b(c^2+a^2)} = \dfrac{c\gamma}{-a^2b^2}$.

The circle through DEF has for its equation

$$a\beta\gamma+\ldots+=(a\alpha+\ldots+\ldots)k\left[bc\omega_a\alpha+\ldots+\ldots\right]/\lambda^4 \quad \text{.....(i.)},$$

where $\lambda^2 \equiv a^2b^2+b^2c^2+c^2a^2$.

In my note on "A Property of the Circumcircle" (*Proc. Lond. Math. Soc.*, Vol. xxii., pp. 470–2), I have shown that this circle (i.) is the image of the circumcircle with regard to $\Omega\Omega'$, and that it has the properties mentioned in the question.

---

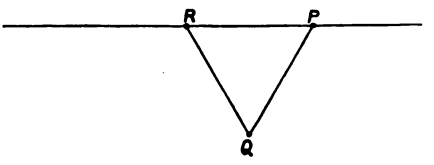
**7826.** (Prof. Orchard, B.Sc., M.A.)—P, Q, R are three points in a plane; PQ = QR, and R is fixed. P, Q rotate uniformly round Q, R respectively, and PQ, QR separate from a fixed common chord at equal

Digitized by Google

rates and in opposite directions. Show that the motion of P relative to R is oscillatory.

*Solution by* H. J. WOODALL.

I assume that "PQ, QR separate from a fixed common chord at equal rates and in opposite directions" means that PQ and QR make equal angles with this fixed chord. Hence $\angle QRP = \angle QPR$; but $QR = QP$; hence P moves on this "fixed common chord," and its motion is oscillatory, and double that of the projection of Q on this fixed common chord.




**11426.** (Col. H. NUTTALL.)—In the *Monist Quarterly Magazine*, of January, 1891, in an article by CHARLES S. PIERCE, entitled "The Architecture of Theories," he states that "within another century our grandchildren will surely know whether the three angles of a triangle are greater or less than 180°; that they are exactly that amount is what nobody ever can be justified in concluding. It is true that according to the axioms of geometry the sum of the three angles of a triangle is precisely 180°; but these axioms are now exploded, and geometers confess that they, as geometers, know not the slightest reason for supposing them to be precisely true." Required to know whether the axioms of geometry *are* exploded as here stated.

*Solution by* H. J. WOODALL.

It appears to be Mr. PIERCE'S wish to throw round the axioms of Geometry a nebulosity something similar to that which already envelopes so-called Philosophy. His attempt does not deserve success, nor will he attain to it unless he keeps his mind clear on the subject. Would it be presumptuous to suggest to him that he has confounded two very different things: (1) Euclidian geometry, or geometry on a true plane; (2) Non-Euclidian geometry, which is geometry as it has been conceived by such master minds as Clifford, Lobatchewsky, &c.? The axioms of geometry rest, however, on firmer ground than mere assertion. The doubt he has suggested may have arisen in a similar manner to that which suggests a conceivable impossibility of *exact* construction of a hexagon of given length of side. Both, however, reduce to the theorem of the equality of alternate and opposite angles made by a straight line on two parallel lines, a doubt which may be at once removed by conception of superposition.

Digitized by Google

**11371.** (Professor MINCHIN, M.A.)—The axes of three coplanar screws, of pitches $p_\alpha$, $p_\beta$, $p_\gamma$, form a triangle whose sides are $a$, $b$, $c$, respectively; prove that the pitches of the two screws (other than that perpendicular to their plane) that can be drawn reciprocal to them through any point O in their plane, are the roots of the equation

$$\frac{ap}{x+p_\alpha}+\frac{bq}{x+p_\beta}+\frac{cr}{x+p_\gamma}=0,$$

where $p$, $q$, $r$ are the perpendiculars from O on $a$, $b$, $c$ (reckoned all positive when O is inside the triangle).

*Solution by* Professor ANDERSON, M.A.

Let OX be the axis of a screw reciprocal to the three given screws whose axes are AB, BC, and CA. Let the direction OX make angles $\alpha$, $\beta$, $\gamma$ with the directions BC, CA, AB; and let the angles which planes drawn through OX parallel to BC, CA, AB make with the plane ABC be denoted by $\xi$, $\eta$, $\zeta$.

The lengths of the shortest distances between OX and BC, CA, AB are $p\sin\xi$, $q\sin\eta$, $r\sin\zeta$; hence we have the conditions

$$(p_\alpha+x)\cos\alpha+p\sin\xi\sin\alpha=0,\quad (p_\beta+x)\cos\beta+q\sin\eta\sin\beta=0,$$
$$(p_\gamma+x)\cos\gamma+r\sin\zeta\sin\gamma=0.$$

It is easily shown, by spherical trigonometry, that

$$\sin\xi\sin\alpha=\sin\eta\sin\beta=\sin\zeta\sin\gamma,$$

and that
$$\sin A\cos\alpha+\sin B\cos\beta+\sin C\cos\gamma=0;$$

hence
$$\frac{ap}{x+p_\alpha}+\frac{bq}{x+p_\beta}+\frac{cr}{x+p_\gamma}=0.$$

**11062.** (D. BIDDLE.)—The ten figures, 0 ... 9, are arranged at random in two rows. Find the probability that $n$ (and only $n$) of the five lower ones are less than the several figures immediately above them.

*Solution by* H. W. CURJEL, B.A.; Prof. RADHIKANSHUAN; *and others.*

Let $a$ be a figure in the top row, and $b$ be the one below it. An arrangement in which all the other figures are in the same places, but $a$ and $b$ are interchanged, is equally likely. Hence, since $a$ is not equal to $b$, the chance that $a$ is greater than $b$ is $\frac{1}{2}$. Hence the required chance is

$$(\tfrac{1}{2})^5\,5!/n!\,(5-n)!.$$

**11099.** (Professor GAY.)—Par quatre points fixes C, D, E, F, pris dans un ordre quelconque sur une ellipse donnée, on fait passer des parallèles qui

Digitized by Google

coupent la courbe en quatre autres points variables C′, D′, E′, F′ ; trouver (1) les lieux des points d'intersection des droites C′D′ et E′F′, D′C′ et E′F′, E′C′ et D′F′; (2) le lieu d'un point P de la droite C′D′ tel que PC′ : PD′ = K, K étant constant ; et (3) l'enveloppe de cette droite.

*Solution by* Professor SCHOUTE.

Projetons la figure par des rayons parallèles, de manière que la projection de l'ellipse soit un cercle. Il est évident que par rapport à cette projection les trois lieux de (1), le lieu de (2), et l'enveloppe de (3) sont des circonférences de cercle concentriques à la projection de l'ellipse. Donc ces lieux et cette enveloppe sont des ellipses concentriques et homothètiques à l'ellipse donnée dans la figure originale.

**11452.** (Professor DE LONGCHAMPS.)—Résoudre les équations $(y-z+b^2-c^2)^2 = 4a^2x$, $(z-x+c^2-a^2)^2 = 4b^2y$, $(x-y+a^2-b^2)^2 = 4c^2z$.

*Solution by* R. F. DAVIS, M.A.; W. J. GREENSTREET, M.A.; *and others.*

Take the square roots of each member of the three given equations and add ; thus $ax^{\frac{1}{2}} + by^{\frac{1}{2}} + cz^{\frac{1}{2}} = 0$, or $a^2x = (by^{\frac{1}{2}} + cz^{\frac{1}{2}})^2$.

Substituting in first equation and reducing

$$\{(y+z)-(b^2+c^2)\}^2 = 4(y^{\frac{1}{2}}z^{\frac{1}{2}} + bc)^2, \quad y+z-(b^2+c^2) = -2(y^{\frac{1}{2}}z^{\frac{1}{2}} + bc),$$

$$y^{\frac{1}{2}} + z^{\frac{1}{2}} = b-c.$$

Thus $x = (b-c)^2$, $y =$ &c., $z =$ &c.

**11392.** (E. M. LANGLEY, M.A.)—SY is drawn perpendicular to the tangent at P to a conic from the focus S. The line joining P to the vertex A meets the corresponding directrix in E. Show that AY is a median of the triangle SYE.

*Solution by* R. KNOWLES, B.A.; Professor BHATTACHARYA; *and others.*

Let $x_1$, $y_1$ be the coordinates of P ; the equation to AY is of the form

$$a^2y_1y + b^2x_1x - a^2b^2 = \lambda(b^2x_1y - a^2y_1x + a^2y_1e) ;$$

put $x = a$, $y = 0$, and it becomes

$$a^2y_1^2y + b^2x_1y_1x - a^2b^2y_1 = (1+e)(a-x_1)(b^2x_1y - a^2y_1x + a^2y_1e) \quad \ldots\ldots(1) ;$$

the equation to PA is $y(x_1-a) = y_1(x-a)$, put $x = a/e$, then the coordinates of the mid-point of SE are $x = a(1+e^2)/2e$, $y = ay_1(1-e)/2e(x_1-a)$, and these values satisfy (1) ; therefore AY is a median of the triangle SYE.

Digitized by Google

# APPENDIX.

## UNSOLVED QUESTIONS.

6497. (Dr. Hopkinson, F.R.S.)—Assuming the equations of motion of the ether in a transparent body to be

$$\left.\begin{aligned}\frac{d^2\eta}{dt^2} &= a_1\frac{d^2\eta}{dx^2} + a_2\frac{d^4\eta}{dx^4} + \ldots + k\frac{d}{dt}\left(\frac{d\zeta}{dx^2}\right)\\ \frac{d^2\zeta}{dt^2} &= a_1\frac{d^2\zeta}{dx^2} + a_2\frac{d^4\zeta}{dx^4} + \ldots - k\frac{d}{dt}\left(\frac{d^2\eta}{dx^2}\right)\end{aligned}\right\};$$

(1) when $k = 0$, find the refractive index for a given ray; (2) when $k$ is small but not zero, show that the rotatory power for a ray, whose wave-length *in vacuo* is $\lambda$, varies as $\mu^2\frac{d}{d\lambda}\left(\frac{\mu}{\lambda}\right)$.

6505. (Professor Cremona.)—Les plans qui coupent en quatre points harmoniques une courbe gauche de 4e ordre et 2e espèce (courbe gauche unicursale de 4e ordre, sans points doubles) enveloppent une surface de Steiner pour laquelle la courbe donnée est asymptotique.

6508. (Rev. Dr. Salmon, F.R.S.) — Find (1) the degree of the evolute of a circular cubic or bi-circular quartic; and (2) how the characteristics of the evolute are affected when the curve is circular or bicircular.

6510. (Sir R. Stawell Ball, F.R.S.)—If a rigid body can be rotated about three lines in space, prove (1) that it can be screwed along the three axes of the hyperboloid containing those lines; and (2) show that the pitches of the three screws are inversely proportional to the squares of the axis.

6511. (Professor Burnside, M.A.) — Determine a property of the surface which renders the double integral

$$\iint\left\{N\left(\frac{1}{R}+\frac{1}{R_1}\right)-\frac{P^2}{RR_1}\right\}dx\,dy$$

a maximum, where N is the normal, P the perpendicular from the origin on the tangent plane, and R, $R_1$ the principal radii of curvature.

6513. (Professor Minchin, M.A.)—A plane curve rolls without sliding, with an angular velocity varying in any way, along a fixed plane curve; prove that the acceleration of the point of contact, considered not as a



Digitized by Google

point in fixed space, but as a point of the rolling curve, is at any instant $\frac{\omega^2}{1/\rho \pm 1/\rho'}$, and show how to find the successive time rates of increase of the components of acceleration of this point parallel to the tangent and normal.

6518. (The late C. J. Monro, M.A.)—Is it two to one that two chords meet within a circle, when each is determined by passing through two random points? [See Vol. VII., pp. 20, 51.]

6520. (R. Tucker, M.A.)—A cube is cut into two parts by a plane the trace upon which is a regular hexagon; find position of centre of inertia of one of the parts.

6521. (The late T. Cotterill, M.A.)—Conicoids circumscribing a quartic curve in space have a common self-conjugate tetrahedron. Show that a plane cuts the quartic in four points, and the tetrahedron in four lines, such that triangles can be found circumscribing any conic inscribed in the four lines conjugate to any conic through the four points.

6557. (J. W. Russell, M.A.)—An ellipse is set rotating about a diameter inclined at an angle $\alpha$ to the major axis, with angular velocity $\Omega$; show that the angular velocities about the major and minor axes will oscillate between the respective limits $\Omega \cos\alpha$, $\Omega$ and $\Omega \sin\alpha$, $-\Omega \sin\alpha$; and that ($a$ and $b$ being the semi-axes), the period of the complete oscillation is

$$\frac{4}{\Omega}\left(\frac{a^2+b^2}{a^2-b^2}\right)^{\frac{1}{2}} \int_0^{\frac{1}{2}\pi} \frac{d\phi}{(1-\sin^2\alpha \sin^2\phi)^{\frac{1}{2}}}.$$

6566. (The late Professor Wolstenholme, Sc.D.)—The tangents at A, B, C to the circle ABC form a triangle A′B′C′; AA′, BB′, CC′ concur in O′, and O is the centre of the circle ABC; prove that

$$OO' = R\left\{\frac{1-\cos(B-C)\cos(C-A)\cos(A-B)+\cos A\cos B\cos C}{2(1+\cos A\cos B\cos C)} \right.$$
$$\left. \frac{+\cos 2A\cos 2B\cos 2C}{2(1+\cos A\cos B\cos C)}\right\}^{\frac{1}{2}}.$$

6582. (R. A. Roberts, M.A.)—If a bicircular quartic meet a conic, show that the sum of the eccentric angles of the eight points of intersection is zero.

6592. (J. W. Russell, M.A.)—Show that

$$\int_0^1 \int_0^{2\pi} \int_{-\infty}^{+\infty} \frac{(az^2\cos\phi - x_0 z)\,dz\,d\phi\,du}{[(az\cos\phi - x_0)^2 + (bz\sin\phi - y_0)^2 + u^2]^{\frac{3}{2}}}$$
$$= -\frac{4bx_0}{a}\int_0^{\pi} \frac{\cos^2\theta\,d\theta}{a^2\sin^2\theta + b^2\cos^2\theta}, \quad \text{provided} \quad \frac{x_0^2}{y^2} + \frac{y_0^2}{b^2} < \text{or} = 1.$$

6593. (C. Leudesdorf, M.A.)—Defining a self-conjugate pentagon with regard to a conic as being such that each of its vertices is the pole of the side opposite to it, show that, if two conics $S = 0$, $S' = 0$ are such that pentagons self-conjugate with regard to the latter can be inscribed in the former, then $\Theta'(\Theta'^2 - 4\Delta'\Theta) + 4\Delta\Delta'^2 = 0$.

Digitized by Google

6609. (J. Hammond, M.A.)—Find the approximate value of the following series, and its equivalent integral :—

$$\frac{1}{2!}+\frac{1}{3!}(1+\tfrac{1}{2})+\frac{1}{4!}(1+\tfrac{1}{2}+\tfrac{1}{3})+\frac{1}{5!}(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4})+\ldots$$

$$\equiv\int_0^1 e^x \log\left(\frac{x}{1-x}\right)dx.$$

6618. (J. W. Russell, M.A.)—Show that the six anharmonic ratios of the four lines $ax^4+4bx^3y+6cx^2y^2+4dxy^3+ey^4=0$ are the roots of the equation in $z$,

$$\frac{1}{\theta^2}-\frac{1}{\theta^3}=\frac{4}{27}-\frac{4T^2}{S^3},$$

where $\theta=z+\frac{1}{Z}-1$ and T are the invariants of the given quantic.

6621. (W. R. Westropp Roberts, M.A.)—A heavy double ladder in the form of an inverted V is placed on a rough horizontal plane, and moves under the influence of gravity; show that the angular velocity at any time can be expressed in finite terms of the height of the centre of gravity, if $\frac{n}{(n^2-1)^{\frac{1}{2}}}$ is an integer, where $n=\frac{a\mu}{k^2+a^2}$.

7145. (The late Prof. Wolstenholme, Sc.D.)—A triangle $\alpha\beta\gamma$ of given form circumscribes a given triangle ABC, and O is the common point of the circles $\alpha$BC, $\beta$CA, $\gamma$AB; a triangle $abc$ is inscribed in ABC having its sides parallel to those of $\alpha\beta\gamma$, and O$\alpha$, O$\beta$, O$\gamma$ meet the sides of ABC in $\alpha'$, $\beta'$, $\gamma'$: prove that (1) the envelopes of $bc$, $ca$, $ab$ are parabolas having a common focus O′, such that O, O′ are foci of a conic inscribed in ABC; and the points of contact are similar points of these parabolas; (2) the envelopes of $\beta'\gamma'$, $\gamma'\alpha'$, $\alpha'\beta'$ are conics having a common focus O′, each touching two sides of the triangle ABC, and having one real common tangent; (3) the point of concurrence of $a\alpha$, $b\beta$, $c\gamma$ is an hyperbola having one vertex at O, and its transverse axis along OO′.

7162. (H. L. Orchard, M.A.)—ABC is a "perfectly rough" inclined plane. When AC is base a sphere rolls down in the same time that a cylinder does when AB is base. Find the angle of the plane.

7163. (Asutosh Mukhopâdhyây, B.A., F.R.A.S.)—Find the mutual action of two indefinitely small plane closed voltaic currents.

7165. (The late Prof. Wolstenholme, Sc.D.)—The pedal of an ellipse being taken with respect to a point O between its foci, prove that (1) the two real foci of the pedal (besides the double focus O) will lie on a similar ellipse whose vertices are the feet of the directrices of the given ellipse, and will be the feet of normals drawn from O; (2) the vector equation of the pedal will be $2r_1=e(r_2+r_3)$, where $e$ is the excentricity of the given ellipse; (3) the same results will be true for an hyperbola when O is a point on the transverse axis not between the foci; (4) when O is a point on the conjugate axis, the two foci of the pedal are the feet of normals drawn from O to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{b^2}{a^2+b^2}$, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ being the given hyperbola, and the vector equation is $2br_1=(a^2+b^2)^{\frac{1}{2}}(r_2\pm r_3)$;

Digitized by Google

(5) when the pedal of the ellipse $b^2x^2+a^2y^2=a^2b^2$ is taken for a point O on the transverse axis but not between the foci, the other real foci of the pedal lie on the transverse axis at distances $\frac{h}{a^2-b^2}[hb \pm a(h^2-a^2+b^2)^{\frac{1}{2}}]$, and the vector equation is

$$[bh+a(h^2-a^2+b^2)^{\frac{1}{2}}]r_2+[bh-a(h^2-a^2+b^2)^{\frac{1}{2}}]r_3=2abr_1,$$

$h$ being the distance of O from the centre; (6) the results for an hyperbola when O lies betwen the foci, or for an ellipse when O lies on the minor axis, may be inferred from this.

7177. (J. Hammond, M.A.)—Prove that, if $\phi(x)=\phi\left(\frac{cx}{1-x}\right)$,

$$\int_0^\infty \frac{\phi(x)}{x^2}dx=\frac{1}{c}\int_0^1 \phi(x)\frac{dx}{x^2}, \qquad \int_0^\infty \frac{\phi(x)\,dx}{x(x+c-1)}=\int_0^1 \frac{\phi(x)\,dx}{x(x+c-1)}.$$

7178. (W. J. C. Sharp, M.A.)—If three concyclic foci of a bicircular quartic, or circular cubic, be given, and also a tangent and its point of contact, determine the curve.

7182. (R. Knowles, B.A., L.C.P.)—Solve the symbolical equations

$$Du+(D-n-1)^2\epsilon^x u=0.$$

7183. (Asutosh Mukhopâdhyây, B.A.)—A material point is so constrained in its motion that its velocity always $=u+\frac{pv\cos\alpha+v\sin\alpha}{(1+p^2)^{\frac{1}{2}}}$, where $u>v$; find its path from one given point on the axis of $x$ to another when the time the journey takes is a minimum.

7184. (D. Edwardes.)—Prove that

$$I\equiv\int_0^x\int_0^y (x-y)^{\frac{1}{2}(n-2)}(y-z)^{\frac{1}{2}(n-2)}f^n(z)\,dy\,dz$$
$$=[\Gamma(\tfrac{1}{2}n)]\,.\,[\text{remainder after } n \text{ terms of expansion of } f(x)],$$

$n$ being an integer and $f^n(z)$ denoting the $n$th differential coefficient of $f(z)$.

7193. (The late Professor Wolstenholme, Sc.D.)—Tangents are drawn from the point $(c, 0)$ to the curve $r^n\cos n\alpha=c^n\cos n(\theta+\alpha)$; prove that their points of contact lie on a curve of the $(n-1)^{\text{th}}$ degree.

7195. (Professor Hudson, M.A.) — Two equal perfectly elastic spherical particles are projected in a vertical plane from two points in the same horizontal line, one vertically upwards, the other in such a manner as to impinge obliquely beneath the former with a velocity $v$ at an angle $45°$, when both are at their greatest height: find (1) the distance between the two points at which they again meet the horizontal line; and (2) the first particle falls at the same place at which the second would have fallen had there been no impact: prove that $2v^2=9gh$, where $h$ is the height of the point of impact.

7198. (Elizabeth Blackwood.) —In a circle two diameters are drawn at right angles to each other; in these diameters four points P, $p$, Q, $q$ are taken at random; find the chance (1) that the straight lines $Pp$, $Qq$ will intersect within the circle, (2) that they will coincide.

Digitized by Google

7199. (Rev. T. P. Kirkman, M.A., F.R.S.)—One or more 6-partitions of any number $n > 8$ can be written in order $a_1 + a_2 + \ldots + a_6 = n$ $(a_r \geqq 1)$ on the sides of a 6-gon, such that, if on the side carrying $a_r$ a $(3+r)$-gon be drawn, the figure can be completed by pentagons only into a polyedron having only triad summits. The proposition is true when for 6 we read 5 or 4, and for "any number > 8," "demanded numbers > 9," the word pentagons being retained. It is not true if for 6 we read 7 or more.

7202. (Captain P. A. MacMahon, B.A.)—The locus of centres in Quest. 7172 being the straight line through the nine-point centre at right angles to the line joining that centre with the orthocentre of the triangle, find the envelope of this line, being given the base and vertical angle of a triangle. Note the case in which the envelope reduces to a point.

7206. (D. Edwardes.)—Prove that

$$\int_0^K \frac{(\mathrm{cn}^2 u + k'^4 \mathrm{sn}^2 u)^{\frac{1}{2}}}{\mathrm{dn}^2 u}\, du = \frac{1}{k'} \mathrm{E}^1(k),$$

where $k'^2 = 1 - k^2$, and $\mathrm{E}^1(k)$ denotes the complete elliptic integral of the second kind whose modulus is $k$.

7207. (Professor Orchard, M.A.)—A fixed circle passes through the centre of the ellipse $r = \dfrac{l}{1 + e\cos\theta}$, and has the same area. The ellipse revolves round an axis through its centre perpendicular to its plane. Find, for a single revolution, the area common to the two curves.

7210. (R. Knowles, B.A., L.C.P.)—Show (1) that

$$\frac{x}{n} + \frac{x^2}{n+1} + \frac{x^3}{n+2}\cdot\frac{1}{1\,.\,2} + \frac{x^4}{n+3}\cdot\frac{1}{1\,.\,2\,.\,3} + \text{\&c. ad inf.}$$
$$\equiv \epsilon^x\left[1 - (n-1)\,x^{-1} + (n-1)\,(n-2)\,x^{-2} \ldots \pm (n-1)\,(n-2) \ldots 2\,.\,1\,x^{-(n-1)}\right]$$
$$\mp (n-1)\,(n-2) \ldots 2\,.\,1\,x^{-(n-1)};$$

and (2) find what the relation will be if $n = a \div b$.

[From the above many series may be deduced, whose limiting ratios are equal; thus, *ex. gr.*, (1) if $(n = 2,\ x = 1)$, and (2) if $(n = 3,\ x = 1)$, we have, respectively, (3) $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \frac{1}{30} + \text{\&c. ad inf.} \equiv 1$ ..... (1),

$\frac{1}{3} + \frac{1}{4} + \frac{1}{10} + \frac{1}{35} + \frac{1}{105} + \text{\&c.} \equiv \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \ldots\ldots$ (2).

7211. (Professor Asutosh Mukhopâdhyây, M.A.) — Can $(-1)^{m/n}$ be real for any values of $h$, $m$, $n$? If so, discriminate the cases.

7214. (R. Tucker, M.A.)—ABC is a triangle in a circle. A′, B′, C′, A″, B″, C″ are the mid-points respectively of the arcs BC, CA, AB, B′C′, A′A′, A′B′; prove that the pedal lines of A″, B″, C″ with respect to ABC pass through a point.

7293. (C. Leudesdorf, M.A.) — Given two circles, whose centres are B and C, and a rectangular hyperbola which has its transverse axis along BC and which passes through the two finite points common to the circles: show that, if a conic be drawn to touch the hyperbola and to have double contact with each of the circles, the distance between its chords of contact with these is $2AB\,.\,BC \div BC$, where A is the reflection, with regard to the radical axis of the circles, of the centre of the hyperbola.

Digitized by Google

7295. (G. S. Carr, B.A.) — Let $V = f(x, y, z, p, q, r, s, t)$, where $z$ is a function of $x$ and $y$, and $p, q, r, s, t$ have the usual meanings. Let the *form* of $z$ vary; and let $\frac{dV}{dz}$, $\frac{dV}{dp}$, &c. be denoted by Z, P, Q, R, S, T;

$$\left(P - \frac{dR}{dx} - \tfrac{1}{2}\frac{dS}{dy}\right)\delta z + \tfrac{1}{2}S\delta q + R\delta p \text{ by } \phi,$$

$$\left(Q - \frac{dT}{dy} - \tfrac{1}{2}\frac{dS}{dx}\right)\delta z + \tfrac{1}{2}S\delta p + T\delta q \text{ by } \psi,$$

and

$$\left(Z - \frac{dP}{dx} - \frac{dQ}{dy} + \frac{d^2R}{dx^2} + \frac{d^2S}{dx\,dy} + \frac{d^2T}{dy^2}\right)dz \text{ by } \chi.$$

Then $\delta V = \frac{d\phi}{dx} + \frac{d\phi}{dy} + \chi$; and thence, $\delta \int_{x_0}^{x_1}\int_{y_0}^{y_1} V\,dx\,dy$

$$= \int_{x_0}^{x_1}\left\{[\psi]_{y=y_0}^{y=y_1} + \phi_{y=y_0}\frac{dy_0}{dx} - \phi_{y=y_1}\frac{dy_1}{dx}\right\}dx + \left[\int_{y_0}^{y_1}\phi\,dy\right]_{x=x_0}^{x=x_1} + \int_{x_0}^{x_1}\int_{y_0}^{y_1}\chi\,dx\,dy.$$

[See Jellett's *Calculus of Variations*, p. 227.]

7296. (T. Muir, M.A., F.R.S.E.) — Show that, if $\Omega$ denote the bordered Hessian $\begin{vmatrix} 0 & a & b & c \\ a & P & Q & S \\ b & Q & R & T \\ c & S & T & U \end{vmatrix}$, (1) the equation

$$\left(\frac{d\Omega}{dT}x - \frac{d\Omega}{dS}y\right) = 4c^2\Omega,$$

denotes two straight lines passing through the two points in which the conic $Px^2 + 2Qxy + Ry^2 + 2Sx + 2Ty + V = 0$ is cut by the straight line $ax + by + c = 0$; and (2) the equation of the straight line which completes the inscribed quadrilateral of which the said three lines are sides is

$$\begin{vmatrix} 0 & a & b & c \\ a & P & Q & S \\ b & Q & R & T \\ c & S & T & \frac{2cS}{a} \end{vmatrix} ax + \begin{vmatrix} 0 & a & b & c \\ a & P & Q & S \\ b & Q & R & T \\ c & S & T & \frac{2cT}{b} \end{vmatrix} by - \begin{vmatrix} 0 & a & b & c \\ a & P & Q & S \\ b & Q & R & T \\ c & S & T & 0 \end{vmatrix} c = 0.$$

7297. (E. B. Elliott, M.A.)—If $(x, y, z)$, the type-point of a surface, have its coordinates expressed in terms of the two variable parameters $c, c'$, and if $u, v, w$ denote respectively

$$\frac{dx}{dc}\cdot\frac{dr}{dc'} - \frac{dx}{dc'}\cdot\frac{dr}{dc}, \quad \frac{dy}{dc}\cdot\frac{dr}{dc'} - \frac{dy}{dc'}\cdot\frac{dr}{dc}, \quad \frac{dz}{dc}\cdot\frac{dr}{dc'} - \frac{dz}{dc'}\cdot\frac{dr}{dc},$$

where $r^2 \equiv x^2 + y^2 + z^2$, prove that the point $(x, y, z)$ will be in general an umbilic on the surface in case $\begin{vmatrix} u, & v, & w \\ \frac{du}{dc}, & \frac{dv}{dc}, & \frac{dw}{dc} \\ \frac{du}{dc'}, & \frac{dv}{dc'}, & \frac{dw}{dc'} \end{vmatrix} = 0.$

Digitized by Google

7306. (Professor Hudson, M.A.)— From a point P on a parabola, focus S, PM, PN are drawn perpendicular to the directrix and axis; PT, PG are the tangent and normal limited by the axis: what line represents the resultant of forces represented by PM, PT, PS, PN, PG?

7308. (The Editor.)—A rectilinear beam APB, resting on the horizontal plane AQC at the point A, is supported in a given position by a prop of given length PQ; find (1) in what position PQ sustains the least pressure, and (2) what force is requisite to keep the beam in its position.

7310. (Rev. T. P. Kirkman, F.R.S.) — From $u_{x+1} = x(u_x + u_{x-1})$, find (1) the integer $u_x$; and (2) give the simple question in Tactic to which $u_x$ is the answer.

7316. (H. L. Orchard, M.A.)—Supposing it possible for the earth to collide with an asteroid, the whole of whose kinetic energy was thus turned into heat, find the heat which might be produced by an asteroid of the mass of one thousand kilogrammes reaching the earth's surface with "the velocity from infinity."

7319. (R. Russell, B.A.)—1. P, Q, R are three points on a conic S; $P_1$, $Q_1$, $R_1$ their corresponding points on a confocal $S_1$; if the tangents at Q and R intersect in $P_1$, then P, $Q_1$, $R_1$ are collinear.

2. Again, if we take any two confocals, whose axes are $(a, b)$ and $(a_1, b_1)$, and denote by S the conic $\frac{x^2}{aa_1} + \frac{y^2}{bb_1} = 1$; then the polar of any point on one confocal is the tangent at the corresponding point on the other.

3. If P, Q be any two points on one conic, $P_1$, $Q_1$ their corresponding points on the other; then, if their tangents at P, $Q_1$ intersect at right angles in R, so also do those at $P_1$, Q intersect in $R_1$, and the chord $PQ_1$ is the polar of R with respect to $S \equiv \frac{x^2}{aa_1} + \frac{y^2}{bb_1} - 1 = 0$.

[By the aid of these principles, some very elegant theorems can be obtained.]

7370. (Professor Asutosh Mukhopâdhyây, M.A.)—The upper extremity of a rod, initially vertical, is projected with a given velocity along a smooth groove, inclined to the horizon at a given angle. Determine the motion of the rod, as well as the time when the lower extremity passes the horizontal line through the point of projection. How would the motion be affected if the rod be supposed to be weightless and a heavy particle placed at the lower extremity?

7373. (D. Edwardes.)—ABCD is a square inscribed in a circle, and P a point on the circumference of the circle. The pedal lines with respect to P are drawn of the triangles formed by the sides and diagonals of ABCD. Prove (1) that the area of the quadrilateral formed by these pedal lines is $\frac{1}{4}(PA \cdot PC + PB \cdot PD)$; (2) its maximum area $\frac{1}{4}AB^2\sqrt{2}$, and (3) the angle between its diagonals is $\sin^{-1}\frac{PL + PM}{AB}$, PL, PM being the perpendiculars from P on the diagonals of the square.

7374. (C. Bickerdike.)— A bucket full of water being suspended from the end of a rope passing over a fixed pulley without weight, its other end being attached to weight just balancing bucket and water,

Digitized by Google

suppose the water be allowed to run from a small orifice ($a$) in its bottom, and the bucket be in the form of a paraboloid, what must be the length of the rope so that the bucket may just reach the pulley the moment it is emptied, and what is the velocity acquired in its ascent?

7375. (R. Knowles.)—Prove that

$$\int_0^{\pi} \log \sin x \frac{dx}{1+\cos^2 x} = \pi \left( \frac{2}{2} \log \tfrac{1}{2} + \log \frac{1+2^{\frac{1}{2}}}{2} \right).$$

7383. (The late Professor Townsend, F.R.S.)—Five spheres in a common space being supposed such that the centres of every four of them determine a tetrahedron self-reciprocal with respect to the fifth, show that the (1) every point in the space returns to its original position after five successive inversions with respect to them all in any order; and (2) if the several inversions be supposed performed in the 120 possible different orders, the entire system, consisting of the original point and its several inverses, determines in the space a group of 16 different points, arranged in pairs of homospherical tetrads in 10 different ways.

7384. (Professor Réalis.) — Étant donnée la série illimitée 7, 13, 25, 43, 67, 97, 133, 175 ..., dont le terme général, celui qui en a $n$ avant lui, est $x_n = 3(n^2+n)+7$, démontrer les propositions suivantes:— (1) sur cinq termes consécutifs, pris à volonté dans la série, un terme est divisible par 5; (2) sur sept termes consécutifs, deux sont divisible par 7; (3) sur treize termes consécutifs, deux sont divisible par 13; (4) aucun terme de la série n'est égal à un cube; (5) une infinité de termes, tels que $x_2 = 25$, $x_{37} = 4225$, &c., sont des carrés divisible par 25; (6) la deuxième et la troisième proposition sont comprises, comme cas particuliers, dans la suivante: si N est un nombre premier, de la forme $6n+1$, sur $n$ termes consécutifs de la série, deux sont divisible par N; (7) on peut affirmer aussi que, à l'exception de 5, aucun nombre premier de la forme $6m-1$ ne peut diviser la série.

7387. (The Rev. T. P. Kirkman, M.A., F.R.S.)—ABCDEFGH are eight polyedra having only triad summits, on which all the faces not 5-gons are on A, $3+a$ 9-gons; on B, $3+b$ 10-gons; on C, $3+c$ 12-gons; on D, $3+d$ 15-gons; on E, $3+e$ 16-gons; on F, $3+f$ 20-gons; on G, $3+g$ 21-gons; and on H, $3+h$ 24-gons. Required the construction and symmetry of the eight, and also of a number $N>1$ of $(10r+2)$-edra, which have $10r$ 5-gons, for every value of $r>1$.

7390. (W. S. McCay, M.A.)—If the straight line joining the feet of perpendiculars on the sides of a triangle ABC from any point on the circumscribed circle cut the sides at angles A′, B′, C′, show that (1) its equation in areal coordinates referred to the triangle is

$$x \cot A' + y \cot B' + z \cot C' = 0;$$

(2) this equation is a cubic in a single parameter (either cot A′, cot B′, or cot C′), and the envelope of the line is therefore a curve of the fourth degree with three cusps; (3) the equation of the line and its first and second differentials are all satisfied when $A'+B'+C' = (2n+1)\frac{1}{2}\pi$, and therefore the tangents at the cusps are inclined to each other at angles of 120° [the angles being measured so that

$$B'-C' = A+\pi, \quad C'-A' = B-\pi, \quad A'-B' = C-\pi].$$

Printed by C. F. Hodgson & Son, 2 Newton Street. Holborn, W.C.

Digitized by Google

# MATHEMATICAL WORKS

PUBLISHED BY

# FRANCIS HODGSON,

89 FARRINGDON STREET, E.C.

---

In 8vo, cloth, lettered.

PROCEEDINGS of the LONDON MATHEMATICAL SOCIETY.

Vol. I., from January 1865 to November 1866, price 10s.
Vol. II., from November 1866 to November 1869, price 16s.
Vol. III., from November 1869 to November 1871, price 20s.
Vol. IV., from November 1871 to November 1873, price 31s. 6d.
Vol. V., from November 1873 to November 1874, price 15s.
Vol. VI., from November 1874 to November 1875, price 21s.
Vol. VII., from November 1875 to November 1876, price 21s.
Vol. VIII., from November 1876 to November 1877, price 21s.
Vol. IX., from November 1877 to November 1878, price 21s.
Vol. X., from November 1878 to November 1879, price 18s.
Vol. XI., from November 1879 to November 1880, price 12s. 6d.
Vol. XII., from November 1880 to November 1881, price 16s.
Vol. XIII. from November 1881 to November 1882, price 18s.
Vol. XIV., from November 1882 to November 1883, price 20s.
Vol. XV., from November 1883 to November 1884, price 20s.
Vol. XVI., from November 1884 to November 1885, price 20s.
Vol. XVII., from November 1885 to November 1886, price 25s.
Vol. XVIII., from November 1886 to November 1887, price 25s.
Vol. XIX., from November 1887 to November 1888, price 30s.
Vol. XX., from November 1888 to November 1889, price 25s.
Vol. XXI., from November 1889 to November 1890, price 25s.
Vol. XXII., from November 1890 to November 1891, price 27s. 6d.

---

In half-yearly Volumes, 8vo, price 6s. 6d. each.
(To Subscribers, price 5s.)

MATHEMATICAL QUESTIONS, with their SOLUTIONS, Reprinted from the EDUCATIONAL TIMES. Edited by W. J. C. MILLER, B.A., Registrar of the General Medical Council.

Of this series fifty-seven volumes have now been published, each volume containing, in addition to the papers and solutions that have appeared in the *Educational Times*, about the same quantity of new articles, and comprising contributions, in all branches of Mathematics, from most of the leading Mathematicians in this and other countries.

New Subscribers may have any of these Volumes at Subscription price.

---

Demy 8vo, price 7s. 6d. Second Edition.
(*Used as the Text-book in the Royal Military Academy, Woolwich.*)

LECTURES on the ELEMENTS of APPLIED MECHANICS. Comprising—(1) Stability of Structures; (2) Strength of Materials. By MORGAN W. CROFTON, F.R.S., late Professor of Mathematics and Mechanics at the Royal Military Academy.

---

Demy 8vo. Price 7s. 6d. Third Edition.
(*Used as the Text-book in the Royal Military Academy, Woolwich.*)

TRACTS ON MECHANICS. In Three Parts—Parts 1 and 2, On the Theory of Work, and Graphical Solution of Statical Problems; by MORGAN W. CROFTON, F.R.S., late Professor of Mathematics and Mechanics at the Royal Military Academy. Part 3, Artillery Machines; by Lieut.-Col. EDGAR KENSINGTON, R.A., Professor of Artillery and formerly Instructor in Mathematics at the Royal Military Academy, Woolwich.

---

Third Edition. Extra fcap. 8vo, price 4s. 6d.

ELEMENTARY MANUAL of COORDINATE GEOMETRY and CONIC SECTIONS. By Rev. J. WHITE, M.A., Head Master of the Royal Naval School, New Cross.

Digitized by Google

Eleventh Edition, small crown 8vo, price 2s. 6d.

## AN INTRODUCTORY COURSE OF PLANE TRIGONOMETRY AND LOGARITHMS.

By JOHN WALMSLEY, B.A.

"This book is carefully done; has full extent of matter, and good store of examples."—*Athenæum.*

"This is a carefully worked out treatise, with a very large collection of well-chosen and well-arranged examples."—*Papers for the Schoolmaster.*

"This is an excellent work. The proofs of the several propositions are distinct, the explanations clear and concise, and the general plan and arrangement accurate and methodical."—*The Museum and English Journal of Education.*

"The explanations of logarithms are remarkably full and clear. . . . The several parts of the subject are, throughout the work, treated according to the most recent and approved methods. . . . It is, in fact, a book for *beginners*, and by far the simplest and most satisfactory work of the kind we have met with."—*Educational Times.*

---

## A KEY

to the above, containing Solutions of all the Examples therein. Price Five Shillings (nett).

---

*By the same Author.*

New Edition, fcap. 8vo, cloth, price 5s.

## PLANE TRIGONOMETRY AND LOGARITHMS.

*FOR SCHOOLS AND COLLEGES.* Comprising the higher branches of the subject not treated in the elementary work.

"This is an expansion of Mr. Walmsley's 'Introductory Course of Plane Trigonometry,' which has been already noticed with commendation in our columns, but so greatly extended as to justify its being regarded as a new work. . . . It was natural that teachers, who had found the elementary parts well done, should have desired a completed treatise on the same lines, and Mr. Walmsley has now put the finishing touches to his conception of how Trigonometry should be taught. There is no perfunctory work manifest in this later growth, and some of the chapters—notably those on the imaginary expression $\sqrt{-1}$, and general proofs of the fundamental formulæ—are especially good. These last deal with a portion of the recent literature connected with the proofs for $\sin(A+B)$, &c., and are supplemented by one or two generalized proofs by Mr. Walmsley himself. We need only further say that the new chapters are quite up to the level of the previous work, and not only evidence great love for the subject, but considerable power in assimilating what has been done, and in representing the results to his readers."—*Educational Times.*

---

IN PREPARATION.

*By the same Author.*

*Suitable for Students preparing for the University Local and similar Examinations.*

## AN INTRODUCTION TO MECHANICS.

With numerous Examples, many of which are fully worked out in illustration of the text.

---

Demy 8vo, price 2s.

## A CHAPTER ON THE INTEGRAL CALCULUS.

By A. G. GREENHILL, M.A.

Crown 8vo, price 1s. 6d.

## THE ELEMENTS OF LOGARITHMS;

with the Sandhurst and Woolwich Questions for 1880-87. By W. GALLATLY, M.A.

Demy 8vo, price 5s.

## ALGEBRA IDENTIFIED WITH GEOMETRY,

in Five Tracts. By ALEXANDER J. ELLIS, F.R.S., F.S.A.

Demy 8vo, sewed, 1s. 4d.

## NOTES ON THE FIRST PRINCIPLES OF DYNAMICS.

By W. H. H. HUDSON, M.A., Professor of Mathematics in King's College, London.

Digitized by Google

Now ready, Royal 8vo, 967 pp., with 285 Diagrams, price 34s.

# A SYNOPSIS

## OF ELEMENTARY RESULTS IN

# PURE MATHEMATICS,

*Containing Six Thousand Propositions, Formulæ, and methods of Analysis, with abridged Demonstrations; supplemented by an Index to the articles on Pure Mathematics which have appeared in thirty-two* BRITISH AND FOREIGN JOURNALS AND TRANSACTIONS *during the present century, containing about* 14,000 *references.*

## BY G. S. CARR, M.A.,

Late Prizeman and Scholar of Gonville and Caius College, Cambridge.

---

The work may also be had in Sections separately, as follows:—

| | | s. | d. |
|---|---|---|---|
| Section | I.—MATHEMATICAL TABLES; including C. G. S. units. Least Factors from 1 to 99,000. The Gamma-function, &c. ..... Price | 2 | 0 |
| „ | II.—ALGEBRA ..... | 2 | 6 |
| „ | III.—THEORY OF EQUATIONS AND DETERMINANTS ..... | 2 | 0 |
| „ | IV.—PLANE TRIGONOMETRY } together; with | | |
| „ | V.—SPHERICAL do. } 22 Diagrams. } ..... | 2 | 0 |
| „ | VI.—ELEMENTARY GEOMETRY; with 47 Diagrams ..... | 2 | 6 |
| „ | VII.—GEOMETRICAL CONICS; with 35 Diagrams ..... | 2 | 0 |
| „ | VIII.—DIFFERENTIAL CALCULUS ..... | 2 | 0 |
| „ | IX.—INTEGRAL CALCULUS ..... | 3 | 6 |
| „ | X.—CALCULUS OF VARIATIONS ..... } | | |
| „ | XI.—DIFFERENTIAL EQUATIONS ..... } | 3 | 6 |
| „ | XII.—CALCULUS OF FINITE DIFFERENCES ..... } | | |
| „ | XIII.—PLANE COORDINATE GEOMETRY, with 182 Diagrams | 6 | 0 |
| „ | XIV.—SOLID COORDINATE GEOMETRY, with 26 Diagrams | 3 | 6 |

N.B.—The Index will not be sold separately, but possessors of previous sections may complete the volume by returning these to the publisher to be bound up with the remaining sections and the Index.

---

The work is in part designed for the use of University and other Candidates who may be reading for examination. It forms a digest of the contents of ordinary treatises and is arranged so as to enable the student rapidly to revise his subjects. To this end all the important propositions in each branch of Mathematics are presented within the compass of a few pages. This has been accomplished, firstly, by the omission of all extraneous matter and redundant explanations, and secondly, by carefully compressing the demonstrations in such a manner as to place only the leading steps of each prominently before the reader. The whole is intended to form a permanent work of reference for mathematicians.

---

*NOTICES OF THE PRESS.*

"The completed work admirably serves the object Mr. Carr set before himself. Every subject that can be classed under the head of Pure Mathematics, with the exception perhaps of Quaternions, appears to us to have been carefully treated on the author's lines."—*The London, Edinburgh, and Dublin Philosophical Magazine.*

"We believe the book to be singularly free from errors. The author is to be heartily congratulated on the successful termination of his labours. The work is printed in a good bold type on good paper, and the figures are admirably drawn."—*Nature.*

"By the boldly printed reference numbers the reader can, if he pleases, go back and back upon the links in the chain of demonstration from the highest theorem to elementary principles. Repetition, otherwise necessary, is thereby avoided, and at the same time the connection of theorems and the course of evolution of formulæ are more clearly exhibited. Although the object of the synopsis is to supplement only, and not to supersede, the usual text-books, the clearness of statement is such that, without any other book, a student who has already some knowledge of mathematics will find it a great saving of time to use this volume as a guide to higher attainments. We cannot adequately set forth in this notice the high estimation we have of this work. The engravings in it are numerous and well executed, especially those referring to geometry and geometrical conics, where a distinguishing thick line is employed with excellent effect. The work is beautifully printed."—*Engineering.*

"The idea of the work is well conceived and is well carried out. In its completed form it is as full, and yet as concise, an index as could be desired of all the most important propositions in the extensive field of Pure Mathematics."—*The Academy.*

Digitized by Google

New and Revised Edition, small crown 8vo, price 2*s*.

# AN INTRODUCTION TO GEOMETRY.

*FOR THE USE OF BEGINNERS.*

CONSISTING OF

## EUCLID'S ELEMENTS, BOOK I.

ACCOMPANIED BY NUMEROUS EXPLANATIONS, QUESTIONS, AND EXERCISES.

BY JOHN WALMSLEY, B.A.

This work is characterised by its abundant materials suitable for the training of pupils in the performance of *original work*. These materials are so graduated and arranged as to be specially suited for class-work. They furnish a copious store of useful examples from which the teacher may readily draw more or less, according to the special needs of his class, and so as to help his own method of instruction.

*OPINIONS OF THE PRESS.*

"We cordially recommend this book. The plan adopted is founded upon a proper appreciation of the soundest principles of teaching. We have not space to give it in detail, but Mr. Walmsley is fully justified in saying that it provides 'for a natural and continuous training to pupils taken in classes.'"—*Athenæum.*

"The book has been carefully written, and will be cordially welcomed by all those who are interested in the best methods of teaching Geometry."—*School Guardian.*

"Mr. Walmsley has made an addition of a novel kind to the many recent works intended to simplify the teaching of the elements of Geometry. . . . The system will undoubtedly help the pupil to a thorough comprehension of his subject."—*School Board Chronicle.*

"When we consider how many teachers of Euclid teach it without intelligence, and then lay the blame on the stupidity of the pupils, we could wish that every young teacher of Euclid, however high he may have been among the Wranglers, would take the trouble to read Mr. Walmsley's book through before he begins to teach the First Book to young boys."—*Journal of Education.*

"We have used the book to the manifest pleasure and interest, as well as progress, of our own students in mathematics ever since it was published, and we have the greatest pleasure in recommending its use to other teachers. The *Questions* and *Exercises* are of incalculable value to the teacher."—*Educational Chronicle.*

KEY to the above, price 3*s*.

---

Just Published, crown 8vo, pp. 114, limp cloth, 1*s*. (Answers, 1*s*.)

MODERN SIDE ARITHMETIC: EXAMPLES. PART I. By the Rev. T. MITCHESON, Assistant-Master in the City of London School. [*Part II. in the Press.*

---

Thirteenth Edition, 12mo, cloth, price 1*s*.

EXAMPLES IN ALGEBRA FOR JUNIOR CLASSES. Adapted to all Text-Books; and arranged to assist both the Tutor and the Pupil.

Third Edition, cloth lettered, 12mo, price 3*s*.

EXAMPLES IN ALGEBRA FOR SENIOR CLASSES. Containing Examples in Fractions, Surds, Equations, Progressions, &c., and Problems of a higher range.

---

Demy 8vo, price 5*s*. each.

TRACTS relating to the MODERN HIGHER MATHEMATICS. By the Rev. W. J. WRIGHT, M.A.

TRACT No. 1.—DETERMINANTS. TRACT No. 2.—TRILINEAR COORDINATES. TRACT No. 3.—INVARIANTS.

The object of this series is to afford to the young student an easy introduction to the study of the higher branches of modern Mathematics. It is proposed to follow the above with Tracts on Theory of Surfaces, Elliptic Integrals and Quaternions.

Digitized by Google

Digitized by Google

Digitized by Google

Digitized by Google

3 2044 102 915 923

Digitized by Google